AN INTRODUCTION OF RESTRICTED LIE ALGEBRAS

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Contents

This survey provides a brief survey of the basic theory of finite flat group schemes and restricted Lie algebras. Particular attention is devoted to the flatness of group schemes, together with examples demonstrating the subtlety of this property and its role in applications.

Throught this survey, S be a locally Noetherian scheme of characteristic $p > 0$. In particular, an S-scheme X/S is finite locally free if and only if it is finite flat.

The main reference for this appendix is [DGA11].

1. Backgrounds on sheaf theory

This section gives a brief discussion on the monomorphism and epimorphism of sheaves, since this is one reason motivating this notes.

Let $\mathcal{C}$ be a site with subcanonical Grothendieck topology and let $\mathrm{Sh}(\mathcal{C})$ be the category of sheaves of sets on $\mathcal{C}$. Having a subcanonical Grothendieck topology means that the embedding

$$h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}), U \mapsto h_U(-) = \mathrm{Hom}(-, U)$$

factors through $\mathrm{Sh}(\mathcal{C})$. So, $\mathcal{C}$ is a fully faithful subcategory of $\mathrm{Sh}(\mathcal{C})$ by Yoneda's lemma. For example, $\mathcal{C}$ can have Zariski, étale, fppf, fpqc topology.

By the general sheaf theory, a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if it is a monomorphism (resp. epimorphism) in category $\mathrm{Sh}(\mathcal{C})$ (see [Sta25, Tag 00WL]).

Let $f : X \to Y$ be a morphism in $\mathcal{C}$.

(1) Then f is a monomorphism in $\mathcal{C}$ if and only h_f it is a monomorphism in $\mathrm{Sh}(\mathcal{C})$ by Yoneda's lemma.
(2) If h_f is an epimorphism in $\mathrm{Sh}(\mathcal{C})$. Then $f : X \to Y$ is an epimorphism in $\mathcal{C}$ again by Yoneda's lemma.
(3) Let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. Suppose f is surjective on the underlying spaces and $f^\#$ is injective, then $(f, f^\#)$ is an epimorphism in the category of schemes just by definition.

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(4) An epimorphism of schemes may fail to be surjective on underlying spaces. Let $Y = \mathrm{Spec}(k[x])$ and $X = \coprod_{a \in k} \mathrm{Spec}(k)$ for some $k = \bar{k}$. Consider the morphism $f : X \to Y$ sending x to a for each $a \in k$. It is not a surjection as it misses the generic point of Y. However, it is an epimorphism in the category of schemes. Indeed, if Z is a scheme with $g, h : Y \to Z$ and $g \circ f = h \circ f$, then g, h agrees on the closed points on Y. Let $Y' = Eq(g, h)$ be the equalizer of Y, which is the maximal locally closed subscheme of Y such that $g_{|Y'} = h_{|Y'}$. By the assumption of g, h, Y' consists of all k-points of Y. But the only subscheme of $Y = \mathbb{A}^1_k$ consists of all closed points is itself. In particular, $g = h$.
(5) The converse is not ture. Let $\mathcal{C} = \mathrm{Sch}_{\mathbb{R}}$ be the large Zariski site over $\mathrm{Spec}(\mathbb{R})$. Let $X = \mathrm{Spec}(\mathbb{C})$ and $Y = \mathrm{Spec}(\mathbb{R})$, $f : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is an epimorphism of schemes since it is both surjective and flat (see [Sta25, Tag 02VW]). But it is not an epimorphism of Zariski sheaves: $\mathrm{Spec}(\mathbb{R})$ has only Zariski cover $\mathrm{Spec}(\mathbb{R})$, so the identity section $id \in h_{\mathbb{R}}(\mathbb{R})$ cannot be the image for some $s \in h_{\mathbb{C}}(\mathrm{Spec}(\mathbb{R}))$, as there is no algebra morphism $\mathbb{C} \to \mathbb{R}$.

The counter-examples above suggests that to have a correspondence of epimorphisms, one need to consider a finer topology and only work on schemes of finite type.

Lemma 1.1 ([Sta25, Lemma 02VW])**.** A surjective and flat morphism of schemes is an epimorphism in the category of schemes.

Proof. This is just remark (3) above, as a flat local homomorphism on local rings is faithfully flat, hence injective. □

Proposition 1.2. Let S be a scheme. If $f : X \to Y$ is flat, locally of finite presentation, and surjective morphism of S-schemes. Then $h_f : h_X \to h_Y$ is an epimorphism in $\mathrm{FPPF}(S)$, the category of sheaves over the big fppf site $(\mathrm{Sch}/S)_{fppf}$.

Proof. The proof is easy. The assumption on f means that $f : X \to Y$ is an fppf cover. Let $T \to Y$ be a morphism, then the base change $T \times_Y X \to T$ is still an fppf cover, as being faithfully flat is fppf local on the base. And the morphism $T \to Y$ comes from $T \times_Y X \to X$ by construction. In particular, this shows $h_X \to h_Y$ is an epimorphism of fppf sheaves. □

Lemma 1.3. If $h_f : h_X \to h_Y$ is an epimorphism in $\mathrm{FPPF}(S)$, then $f : X \to Y$ is surjective.

Proof. Let $y \in Y$ be a point with $y = \mathrm{Spec}(k)$. Since h_f is an epimorphism in $\mathrm{FPPF}(S)$, there exists an fppf covering U of y such that $i : y \to Y$ comes from $j : U \to X$, so y has a preimage in X. □

Remark 1.4. Even in the category of fppf sheaves, being epimorphism of (good) schemes does not force the represented sheaves being epimorphism. The flatness is subtle here. By Lemma 2.2 below, a morphism $f : X \to Y$ between finite flat group schemes gives an epimorphism of fppf sheaves if and only if f is faithfully flat. But an epimorphism of finite flat group schemes can be non-flat: let $S = \mathrm{Spec}(\mathbb{Z})$ and consider the morphism $f : (\mathbb{Z}/2\mathbb{Z})_S \to \mu_{2,S}$ given by

$$f^{\#} : \mathbb{Z}[T, T^{-1}]/(T^2 - 1) \to \mathbb{Z} \times \mathbb{Z}, T \mapsto (1, -1).$$

$f^{\#}$ is an injective ring morphism and f is surjective on underlying spaces. So, f is an epimorphism in the category of schemes.

But $f^{\#}$ fails to be flat: the $\mathbb{F}_2$-fibre $f^{\#} : \mathbb{F}_2[T, T^{-1}]/(T^2 - 1) \to \mathbb{F}_2 \times \mathbb{F}_2, T \mapsto (1, 1)$ is not flat obviously.

2. Finite flat group schemes

Although the previous discussion already provides us what we need, we choose to forget it and write a detailed proof for finite flat group schemes. The reason is that we want to understand the conditions on "flat" and "finite" better.

Let FFGp_S denote the category of commutative finite flat group schemes over S. It is a subcategory of $\mathrm{FPPF}(S)$, the category of sheaves on the big fppf site $(\mathrm{Sch}/S)_{\mathrm{FPPF}}$ of S. An exact sequence in FFGp_S

$$G \to G' \to G''$$

is defined to be an exact sequence of fppf sheaves.

Let $G, G' \in \mathrm{FFGp}_S$ be two commutative finite flat group schemes and let $\phi : G \to G'$ be a (group scheme) morphism. We say ϕ is injective (resp. surjective) if it is a monomorphism (resp. epimorphism) in the category of fppf sheaves, i.e., the sequence $0 \to G \to G'$ (resp. $G \to G' \to 0$) is exact viewed as a sequence of fppf sheaves.

Lemma 2.1. Let $\phi : G \to G'$ be a morphism in the category FFGp_S, then ϕ is a finite morphism.

Proof. See [Sta25, Tag 035D]. □

Lemma 2.2. Let $\phi : G \to G'$ be a morphism in the category FFGp_S.
(1) ϕ is injective if and only if ϕ is a closed immersion.
(2) If ϕ is flat, $\mathrm{Ker}(\phi) \in \mathrm{FFGp}_S$.
(3) ϕ is surjective if and only if ϕ is faithfully flat.
(4) If ϕ is a closed immersion, $G'/G \in \mathrm{FFGp}_S$.
(5) ϕ is surjective if and only if ϕ is a fppf cover.

Proof. (1). ϕ is monomorphism as fppf sheaves if and only if it is a monomorphism as schemes as we discussed before. As G, G' are assumed to be finite over S, the first statement follows from [Sta25, Tag 04XV].

(2). Let $e_{G'} : S \to G'$ be the unit of G'. Then $\mathrm{Ker}(\phi) := e_{G'} \times_{G'} G$ is always a subgroup scheme of G. It is finite and flat over S by the assumption on G, G' and ϕ.

(3). We first claim that for a general morphism locally of finite presentation $\phi : G \to G'$ between group schemes over S, ϕ is faithfully flat if and only if ϕ is surjective and $\mathrm{Ker}(\phi)$ is flat.

proof of claim. ϕ gives rise to an isomorphism of schemes $\Phi : G \times_{G'} G \cong G \times_S \mathrm{Ker}(f)$, with maps $(g, h) \mapsto (g, g^{-1}h)$ and $(g, k) \mapsto (g, gk)$. If ϕ is faithfully flat, it is an fppf covering in particular. The isomorphism Φ shows that $\phi : G \to H$ becomes a trivial torsor $\phi' : G \times_S \mathrm{Ker}(f) \to G$ after fppf base change. To show ϕ is an epimorphism of fppf sheaves, it suffices to show ϕ' is an epimorphism of fppf sheaves. But this is clear as ϕ' admits a section. And the flatness of $\mathrm{Ker}(\phi)$ is (2).

On the other hand, let ϕ be an epimorphism and suppose $\mathrm{Ker}(\phi)$ is flat. We want to show $\phi : G \to G'$ becomes an fppf $\mathrm{Ker}(\phi)$-torsor. That is, there exists an fppf cover $H \to G'$ such that $H \times_{G'} G \cong \mathrm{Ker}(\phi) \times_S H$.

As ϕ is an epimorphism, $\phi' : H \times_{G'} G \to H$ has a section for some fppf cover $H \to G'$. On the other hand, $H \times_{G'} G \cong H \times_G G \times_{G'} G \cong H \times_S \mathrm{Ker}(\phi)$ is a trivial $\mathrm{Ker}(\phi)$-torsor. So, $\phi : G \to G'$ becomes an fppf $\mathrm{Ker}(\phi)$-torsor. In particular, $\phi' : H \times_{G'} G \to H \cong H \times_S \mathrm{Ker}(\phi) \to H$ is faithfully flat as $\mathrm{Ker}(\phi)$ is assumed to be flat. As being faithfully flat is fppf local on the base, $\phi : G \to G'$ is faithfully flat.

□

Now, it suffices to show that if G, G' are finite flat, then ϕ is surjective implies that it is flat. And the conclusion follows from (2). By [Gro67, Corollary 11.3.11], it suffices to check the flatness fibrewisely. So, we may assume $s = \mathrm{Spec}(k)$ and G_s, G'_s are finite flat group schemes over $\mathrm{Spec}(k)$. And $\phi_s : G_s \to G'_s$ is still surjective as s^{-1} is an exact functor $\mathrm{FPPF}(S) \to \mathrm{FPPF}(s)$. $\mathrm{Ker}(\phi_s)$ is a flat group k-scheme as it is defined over a field. By the criterion before, ϕ_s is faithfully flat and hence flat.

(4). Consider G as a closed subgroup scheme of G'. Then by the general theory of group schemes, the fppf quotient G'/G is represented by a group scheme, which is still denoted by G'/G. The quotient map $\phi : G' \to G'/G$ is faithfully flat, locally of finite presentation and proper by

[BLR90, Theorem 8.12] (since we assume S to be locally Noetherian and $f : G \to S$ is finite flat, $f_*\mathcal{O}_G$ is a finite locally free sheaf over S, making X to be strongly projective over S), making G' an fppf G-torsor over G'/G and such that $G' \times_{(G'/G)} G' \cong G' \times_S G$.

To show G'/G is finite over S, it suffices to show it is proper with finite fibres. Since ϕ is surjective and G' is finite over S, G'/G has to have finite fibres over any point of S. And as G' is finite over S, it suffices to show G'/G is separated over S and locally of finite presentation over S. Since being locally of finite presentation is fppf local on the source and G'/S is finite, it follows that G'/G is locally of finite presentation as $G' \to G'/G$ is an fppf cover.

We now show G'/G is separated. We use H to denote G'/G for convenience. We need to show $\Delta : H \to H \times_S H$ is a closed immersion. As being closed immersion is fppf local on the target and $G' \times_S G' \to H \times_S H$ is an fppf cover, it suffices to show $\Delta' : H \times_{H \times_S H} (G' \times_S G') \to G' \times_S G'$ is a closed immersion. On the other hand, $H \times_{H \times_S H} (G' \times_S G') \cong G' \times_H G' \cong G' \times_S G$. As $\phi : G \to G'$ is a closed immersion, Δ' is a closed immersion.

It remains to show G'/G is flat over S. Since this is a local question, it suffices to show if $A \to B \to C$ is a composition of ring morphisms with C faithfully flat over B and C flat over A, then B flat over A. Let $0 \to M \to M'$ be an exact sequence of A-modules. Since C is flat over A, $0 \to M \otimes_A C \to M' \otimes_A C$ is exact. Since C is faithfully flat over B, $0 \to M \otimes_A B \to M' \otimes_A B$ is exact. So, B is flat over A.

(5). It follows by (3) and Lemma 2.1.

□

Remark 2.3. We remark that for general S, the category FFGp_S is not abelian, as kernel can fail to be flat easily.

3. Restricted Lie algebras

3.1. **Restricted Lie algebras.** Let $\mathcal{F}$ be an fppf contravariant sheaf of commutative groups on S. One can define the Lie algebra $\mathrm{Lie}(\mathcal{F})$ of $\mathcal{F}$ to be the functor

$$T \mapsto \mathrm{Ker}(i^\# : \mathcal{F}(T \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]/(\epsilon^2)) \to \mathcal{F}(T)),$$

where $i : T \to T \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]/\epsilon^2$ is the natural closed immersion. $\mathrm{Lie}(\mathcal{F})$ then becomes a sheaf with a structure of $\mathcal{O}_S$-modules. See [LLR04] for a detailed discussion.

Let $\pi : G \to S$ be a group scheme with unit section $s : S \to G$. Then the Lie algebra $\mathrm{Lie}(G/S)$ of G, viewed as the Lie algebra of the fppf sheaf G on S, coincides with $(s^*\Omega_{G/S})^\vee$. In other words, $\mathrm{Lie}(G/S)$ is the sheaf of invariant vector fields.

Consider the case $G = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are both affine, such that $\mathrm{Lie}(G/S) \cong \mathrm{Der}_A(B, B) \otimes_B A$ and identified with the invariant derivations, i.e., $\Delta \circ D = (1 \otimes D) \circ \Delta$, where Δ is the co-multiplication map. The Lie bracket

$$[-, -] : \mathrm{Lie}(G/S) \times \mathrm{Lie}(G/S) \to \mathrm{Lie}(G/S)$$

is defined to be

$$[D, D'] = DD' - D'D.$$

It is easy to check that the Lie brackets of invariant differentials is still invariant.

And $[-, -]$ is zero if G is commutative. Indeed, $\Delta \circ ([D, D']) = (1 \otimes [D, D']) \circ \Delta = (1 \otimes (DD' - D'D)) \circ \Delta = (D' \otimes D) \circ \Delta - (D \otimes D') \circ \Delta = 0$ as $\iota \circ \Delta = \Delta$ by assumption. Since Δ is injective, $[D, D'] = 0$.

Moreover, a calculation shows that

$$D^n(st) = \sum_{i=0}^{n} \binom{n}{i} D^i(s) D^{n-i}(t).$$

In particular,

$$D^p(st) = D^p(s)t + sD^p(t)$$

if $\mathrm{char}(S) = p > 0$. This gives a p-linear morphism

$$[p] : \mathrm{Lie}(G/S) \to \mathrm{Lie}(G/S).$$

This leads to the following definition.

Definition 3.1 (Restricted Lie algebras). Let R be a commutative ring of characteristic $p > 0$. A restricted Lie algebra over R is a Lie algebra $(L, [-,-])$ consisting of an R-module L and a Lie bracket $[-,-] : L \times L \to L$ together with a p-mapping $(-)^{[p]} : L \to L, x \mapsto x^{[p]}$ such that:

(1) $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}x)^p$, where ad is the adjoint representation of L on itself defined by $\mathrm{ad}(x)(y) = [x, y]$;
(2) $(tx)^{[p]} = t^p x^{[p]}$ for any $t \in R$ and $x \in L$;
(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ for any $x, y \in L$, where $s_i(x, y)$ is $\frac{1}{i}$ times the coefficient of t^{i-1} in the formal expression $(\mathrm{ad}(tx + y))^{p-1}(x)$.

More generally, let S be a scheme of characteristic $p > 0$. A sheaf of restricted Lie algebras over S is a sheaf of $\mathcal{O}_S$-modules $\mathcal{L}$ together with a Lie bracket $[-,-] : \mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{L} \to \mathcal{L}$ and a p-th power map $(-)^{[p]} : \mathcal{L} \to \mathcal{L}$ such that for some affine covering $S = \cup_{i \in I} \mathrm{Spec}(R_i)$, $(\mathcal{L}(\mathrm{Spec}(R_i)), [-,-], (-)^{[p]})$ is a restricted Lie algebra over R_i. We also call it a restricted Lie algebra over S.

A morphism $\phi : \mathcal{L} \to \mathcal{L}'$ of restricted Lie algebras is a morphism of sheaves $\phi : \mathcal{L} \to \mathcal{L}'$ compatible with Lie bracket and p-mapping.

Remark 3.2. (1) Let G/S be a group scheme over a base S of characteristic $p > 0$. $\mathrm{Lie}(G/S)$ then gives a sheaf of restricted Lie algebras over S. And if G is commutative, the Lie bracket is trivial.
(2) Let G/S be a commutative group scheme with locally free Lie algebra. Let F_S be the absolute Frobenius on S and let $F_{G/S} : G \to G^{(p/S)}$ be the relative Frobenius morphism. Then $e^*_{G^{(p/S)}}\Omega_{G^{(p/S)}/S} \cong e^*_{G^{(p/S)}}\mathrm{pr}_1^*\Omega_{G/S} \cong F_S^* e_G^* \Omega_{G/S}$. It follows that $\mathrm{Lie}(G^{(p/S)}/S) \cong F_S^*\mathrm{Lie}(G/S)$. And the p-mapping on $\mathrm{Lie}(G^{(p/S)}/S)$ coincides with the one on $\mathrm{Lie}(G/S)$ pulled back via F_S.

We observe that the Lie algebra of a group scheme only depends on its Frobenius torsion.

Lemma 3.3. Let $F_{G/S} : G \to G^{(p/S)}$ be the Frobenius morphism and let $G[F] \coloneqq \mathrm{Ker}(F)$ be the Frobenius torsion. Then $\mathrm{Lie}(G/S) \cong \mathrm{Lie}(G[F]/S)$.

Proof. By [LLR04, Proposition 1.1], $\mathrm{Lie}(\cdot/S)$ is a left exact functor. Consider the exact sequence

$$0 \to G[F] \to G \xrightarrow{F_{G/S}} G^{(p/S)},$$

which implies an exact sequence of Lie algebras

$$0 \to \mathrm{Lie}(G[F]/S) \to \mathrm{Lie}(G/S) \xrightarrow{\mathrm{Lie}(F_{G/S})} \mathrm{Lie}(G^{(p/S)}/S).$$

$\mathrm{Lie}(F_{G/S})$ is zero obviously. So, $\mathrm{Lie}(G[F]/S) \cong \mathrm{Lie}(G/S)$. $\square$

We say that a group scheme G/S is of height at most one if $F_{G/S} : G \to G^{(p/S)}$ is trivial, i.e., it factors through the unit section $e_{G^{(p/S)}}$.

Theorem 3.4 ([DGA11], Exposé VIIA, Chapter 7). The functor

$$G \mapsto \mathrm{Lie}(G/S)$$

together with a functor

$$\mathcal{L} \mapsto G_p(\mathcal{L})$$

induces an equivalence between the categories of flat locally finitely presented group schemes of height at most one over S and finite locally free restricted Lie algebras over S.

We explain the construction of the functor $G_p(\cdot)$. Let $\mathcal{L}$ be a locally free restricted Lie algebra with Lie bracket $[-,-]$ and p-mapping $[p]$. Let $T\mathcal{L} := \oplus_{n\geq 0}\mathcal{L}^{\otimes n}$ be the tensor algebra of $\mathcal{L}$ and let $U(\mathcal{L}) := T\mathcal{L}/\sim$ be the usual enveloping algebra, where $\sim$ is given by $[x,y] = xy - yx$. Let $J := (x^p - x^{[p]}; x \in \mathcal{L})$ be the two sided ideal in $U(\mathcal{L})$. Then $U_p(\mathcal{L}) := U(\mathcal{L})/J$ has a natural Hopf $\mathcal{O}_S$-algebra structure which is co-commutative. It is commutative if and only if $[-,-]$ is trivial by construction. In particular, the Cartier dual $U_p(\mathcal{L})^D$ is a commutative Hopf $\mathcal{O}_S$-algebra and we can define the group scheme $G_p(\mathcal{L}) := \underline{\mathrm{Spec}}_{\mathcal{O}_S}(U_p(\mathcal{L})^D)$, which is commutative if and only if $[-,-]$ is trivial.

Remark 3.5. We note that a flat locally of finite presentation group scheme of height at most one is actually finite flat. Indeed, as G/S is locally of finite presentation, $F_{G/S} : G \to G^{(p/S)}$ is a finite morphism. As G is of height at most one, $F_{G/S}$ is trivial. So, $G := \mathrm{Ker}(F_{G/S}) = e_{G^{(p/S)}} \times_{G^{(p/S)}} G$ is finite over S. In particular, the category of flat locally finitely presented group schemes of height at most one over S is equivalent to the category of finite flat group schemes of height at most one over S.

Remark 3.6. (1) Let $\pi : G \to S$ be a finite group scheme over S of height at most one. We note that G is flat if and only if $f_*\Omega_{G/S}$ is finite locally free (see [DGA11, Exposé VIIA, Chapter 7]).

(2) In particular, if G is flat over S, then $\mathrm{Lie}(G/S)$ is finite locally free. And if $\mathrm{Lie}(G/S)$ is locally free, G is flat if and only if $G \cong G_p(\mathrm{Lie}(G/S))$.

(3) The converse is not correct: if $\mathrm{Lie}(G/S)$ is locally free, G/S may fail to be flat. Similarly, if G, H are two finite group schemes of height at most one and $\mathrm{Lie}(G/S) \cong \mathrm{Lie}(H/S)$ are isomorphic locally free restricted Lie algebras, then one does not always have $G \cong H$ as group schemes. For example, let $S = \mathbb{P}^1_k$ and $H = \mathrm{Spec}(k) \times_k \mathbb{P}^1_k$ be the trivial group scheme, where $k = \overline{\mathbb{F}}_p$ Let $\mathcal{I}_x \subset \mathcal{O}_S$ be the ideal sheaf of a closed point $x \in S$. Let $G = \underline{\mathrm{Spec}}_{\mathcal{O}_S}(\mathcal{O}_S[T]/(T^p, \mathcal{I}_x T))$ with Hopf structure the same as $\mathbb{G}_a$, which is a finite group scheme over S but is non-flat. It has the structure that being isomorphic to α_p at x and is trivial everywhere else.

Moreover, $f_*\Omega_{G/S} \cong i_*k_x$ is torsion. So, $\mathrm{Lie}(G/S) \cong (f_*\Omega_{G/S})^\vee = 0$ is trivial. In particular, it is locally free.

And as $\mathrm{Lie}(G/S) \cong \mathrm{Lie}(H/S)$, we see $\mathrm{Lie}(\cdot/S)$ is not fully faithful for all fintie group schemes of height at most one.

We now denote by $\mathrm{FFGp}_S^{F\text{-tor}}$ the category of commutative finite flat group schemes of height at most one over S. And let $p\text{-Lie}_{\mathcal{O}_S}$ denote the category of commutative locally free restricted Lie algebras over S. The discussion above implies the following result.

Corollary 3.7. There exists an equivalence of categories

$$\mathrm{Lie}(\cdot/S) : \mathrm{FFGp}_S^{F\text{-tor}} \to p\text{-Lie}_{\mathcal{O}_S}$$

with inverse functor

$$G_p(\cdot) : p\text{-Lie}_{\mathcal{O}_S} \to \mathrm{FFGp}_S^{F\text{-tor}}$$

We now want to give $p\text{-Lie}_{\mathcal{O}_S}$ another description. Namely, if $\mathcal{L}$ is a commutative locally free restricted Lie algebra, then the only additional datum on $\mathcal{L}$ is the p-mapping

$$[p] : \mathcal{L} \to \mathcal{L}$$

such that it is additive p-linear. This is equivalent to an $\mathcal{O}_S$-linear morphism $\phi : F_S^*\mathcal{L} \to \mathcal{L}$ defined by

$$\phi(F_S^* x) = x^{[p]}.$$

In particular, $p\text{-Lie}_{\mathcal{O}_S}$ is equivalent to the category whose objects are pairs

$$\{(\mathcal{L}, \phi); \mathcal{L} \in \mathrm{Vect}_S, \phi \in \mathrm{Hom}_{\mathcal{O}_S}(F_S^*\mathcal{L}, \mathcal{L})\},$$

and obvious morphisms.

For example, if $\mathrm{Lie}(G/S)$ is a locally free Lie algebra for a commutative group scheme, with p-mapping $\phi : F_S^*\mathrm{Lie}(G/S) \to \mathrm{Lie}(G/S)$. Then the p-mapping on $\mathrm{Lie}(G^{(p/S)}/S)$ is given by the pair $(F_S^*\mathrm{Lie}(G/S), F_S^*\phi)$.

Remark 3.8. For a general base scheme S, p-$\mathrm{Lie}_{\mathcal{O}_S}$ is not an abelian category, as the kernel of a morphism of locally free sheaves can fail to be locally free easily.

3.2. **Applications to order p connected group schemes.**

Example 3.9. We use Theorem 3.7 to give a classification of finite flat group schemes of order p and height at most one over a field k of characteristic $p > 0$.

We first notice that for a commutative group scheme $G = \mathrm{Spec}(R)$ over k with co-multiplication Δ, a derivation $D \in \mathrm{Der}_k(R, R)$ is invariant if $\Delta \circ D = (1 \otimes D) \circ \Delta$.

(1) Let $\mu_p = \mathrm{Spec}(k[x, x^{-1}]/(x^p - 1))$ be the multiplicative group of order p. Let $R = k[x, x^{-1}]/(x^p - 1)$. It has co-multiplication $\Delta(x) = x \otimes x$. Then $D = x\frac{dx}{x}$ is an invariant derivation in $\mathrm{Der}_k(R, R)$. The p-mapping is given by $[p](\lambda D) = \lambda^p D^p \equiv \lambda^p D$. So, $\mathrm{Lie}(\mu_p/k) \cong k \cdot D \cong k$, with p-mapping given by $\phi(x) = x^p$.
(2) Let $\alpha_p = \mathrm{Spec}(k[x]/x^p)$ be the additive group of order p. Let $R = k[x]/x^p$. It has co-multiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then $D = \frac{d}{dx}$ is an invariant derivation in $\mathrm{Der}_k(R, R)$. The p-mapping is given by $[p](\lambda D) = \lambda^p D^p = 0$. So, $\mathrm{Lie}(\alpha_p/k) \cong k \cdot D \cong k$, with p-mapping given by $\phi(x) = 0$.
(3) Let (k, ϕ) be a one dimensional restricted Lie algebra over k. $\phi : k \to k$ is an additive p-linear morphism. So, ϕ is determined by $\phi(1) = \lambda \in k$, namely $\phi(x) = \lambda x^p$. If $\lambda = 0$, this corresponds to α_p. If $\lambda \neq 0$, we consider the following diagram

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & ax \arrow[dd, mapsto] \\
& k \arrow[r] \arrow[d, "1"'] & k \arrow[d, "\lambda"] & \\
& k \arrow[r] & k & a^p x^p \lambda \\
x^p \arrow[rrr, mapsto] & & & ax^p
\end{tikzcd}$$

Then the diagram commutes if there exists $a \in k$ such that $a^{p-1} = \lambda$, which gives an isomorphism of restricted Lie algebras with p-mapping given by 1 and λ respectively.

As a corollary, the isomorphism classes of order p finite flat group schemes of height one is characterized by k/k^{p-1}. In particular, if $k = \bar{k}$, then there are only two isomorphism classes μ_p and α_p as shown above.

Also as a corollary, the isomorphism classes of order p multiplicative group (or order p étale group schemes) can be characterized by $k^*/(k^*)^{p-1}$.

Lemma 3.10. A finite flat group scheme G over a perfect field k of characteristic $p > 0$ is of local-local type if and only if G is a successive extension of α_p over k.

Proof. Let p^n be the order of G. By Dieudonné theory, it suffices to show a finite-length Dieudonné module (M, F, V) over $W(k)$ has nilpotent F, V if and only if M has a Dieudonné submodule $(k, 0, 0) \subseteq (M, F, V)$ and

$$(M, F, V)/(k, 0, 0)$$

also has nilpotent F, V.

One direction is clear. Conversely, one can consider the Dieudonné submodule $(\mathrm{Ker}(F), 0, V)$ which is nontrivial as F is nilpotent. Similarly, $(\mathrm{Ker}(F) \cap \mathrm{Ker}(V), 0, 0)$ is a nontrivial Dieudonné submodule of (M, F, V), with $\mathrm{Ker}(F) \cap \mathrm{Ker}(V) \cong k^{\oplus m}$ for some m. This gives the desired submodule $(k, 0, 0) \subseteq (M, F, V)$. The quotient has nilpotent F, V obviously.

□

We say the p-mapping φ on a restricted Lie algebra $\mathcal{L}$ is nilpotent, if

$$(F_S^{n-1})^* \varphi \circ \cdots \varphi : (F^n)^* V \to V$$

is zero for some $n \geq 1$.

Proposition 3.11. Let k be a field of characteristic $p > 0$. Let G be a finite flat commutative group scheme of p-power order over k. Then the p-mapping φ of $\mathrm{Lie}(G/k)$ is nilpotent if and only if G is local-local and φ is bijective if and only if G is local-étale.

Proof. There exists a short exact sequence

$$0 \to G' \to G \to G'' \to 0$$

of finite flat group schemes over k, where G' is local-étale and G'' is local-local. Moreover, this sequence splits if k is perfect. As the Lie algebra is compatible with field extension, we can assume $k = \bar{k}$ is algebraically closed without loss of generality. So, $G \cong G' \times G''$ and $\mathrm{Lie}(G/k) \cong \mathrm{Lie}(G'/k) \oplus \mathrm{Lie}(G''/k)$. It then suffices to show that if G is local-étale, then φ is bijective and that if G is local-local, then φ is nilpotent.

If G is local-étale, G has a decomposition $G \cong \bigoplus_{i \in I} \mu_{p^{n_i}}$ for some index set I, as $k = \bar{k}$. Then the surjectivity follows from the calculation in Example 3.9.

If G is local-local, we may assume $G = G[F]$ of height at most one. It has a successive extension of α_p by Lemma 3.10 as $k = \bar{k}$, which gives $(\mathrm{Lie}(G/k),\ \varphi)$ a successive extension of restricted Lie algebra $(k,\ \varphi = 0)$ by Example 3.9 (see Proposition 4.1 below). In particular, φ is nilpotent.

□

3.3. **Applications to abelian varieties.** An important of the study of restricted Lie algebra is that it leads to the study of the geometric properties of the moduli space of abelian varieties in positive varieties. It is powerful in the sense that it provides a functorial object in positive characteristic so that one can study Arakelov geometry, hyperbolicity, projectivity or moduli spaces, etc.

Lemma 3.12. Let S be a scheme of characteristic $p > 0$. Let $\mathcal{X}/S$ be an abelian scheme, then $\mathcal{X}[F]$ is a finite flat group scheme over S.

Proof. Let $F_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X}^{(p/S)}$ be the relative Frobenius morphism. $F_{\mathcal{X}/S}$ is finite as it is both proper and quasi-finite obviously.

By [Gro67, Corollary 11.3.11] again, it suffices to verify the flatness of $F_{\mathcal{X}/S}$ fibrewisely. Let $F_{A/k} : A \to A^{(p/k)}$ be the relative Frobenius of some abelian variety over k. As it is a finite morphism, the flatness of $F_{A/k}$ comes from the miracle flatness criterion.

□

Proposition 3.13. Let A/k be an abelian variety over a field k of characteristic $p > 0$. Then A is ordinary if and only if the p-mapping on $\mathrm{Lie}(A/k)$ is bijective. And A has p-rank zero if and only if the p-mapping on $\mathrm{Lie}(A/k)$ is nilpotent.

Proof. This is a corollary of Proposition 3.11. □

Corollary 3.14. Let $\mathcal{X}/R$ be a semi-abelian scheme over an equi-characteristic domain (R, m_R) of characteristic $p > 0$. Suppose the generic fibre of $\mathcal{X}$ is an abelian variety of p-rank zero. Then so is any special fibre.

Proof. Let $K = \mathrm{Frac}(R)$ and $X \coloneqq \mathcal{X}_K$ be the generic fibre, which is an abelian variety of charateristic $p > 0$. By the last corollary, $\mathrm{Lie}(X/K)$ has p-mapping nilpotent. Then so is the p-mapping on $\mathrm{Lie}(\mathcal{X}/R) \subset \mathrm{Lie}(X/K)$. And so is the p-mapping on $\mathrm{Lie}(X_k/k)$, where $k = R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. As X_k is semi-abelian, it is an extension of a torus by an abelian variety. But a non-trivial torus will prevent the p-mapping of $\mathrm{Lie}(X_k/k)$ being nilpotent by Proposition 3.11. □

Corollary 3.15. Let A/k be an abelian variety over a field k of characteristic $p > 0$. Then $p\text{-rank}(A) = \dim_k \cap_{n\geq 1}\phi^n(\mathrm{Lie}(A/k))$.

4. Exactness of $\mathrm{Lie}(\cdot/S)$

From now on, by restricted Lie algebras or group schemes we mean commutative restricted Lie algebras or commutative group schemes.

By an exact sequence of (locally free) restricted Lie algebras, we mean a complex in the category of (locally free) restricted Lie algebras such that the sequence of the underlying sheaves is exact.

Proposition 4.1. $\mathrm{Lie}(\cdot/S)$(resp. $G_p(\cdot)$) preserves short exact sequences in the category $\mathrm{FFGp}_S^{F\text{-tor}}$ (resp. $p\text{-Lie}_{\mathcal{O}_S}$).

Proof. Let
$$0 \to G_1 \to G_2 \to G_3 \to 0$$
be an exact sequence of finite flat group schemes of height one over S. $\mathrm{Lie}(\cdot/S)$ is left exact from a general property of the Lie algebra (see [LLR04, Proposition 1.1]). So, we only need to show the right exactness. As $\mathrm{Lie}(G_i/S), i = 1, 2, 3$ are locally free. It suffices to show for any geometric point $s \in S$,
$$0 \to s^*\mathrm{Lie}(G_1/S) \to s^*\mathrm{Lie}(G_2/S) \to s^*\mathrm{Lie}(G_3/S) \to 0$$
is exact by Nakayama's lemma. So, we may assume $S = \mathrm{Spec}(k)$ is a point and $G_i, i = 1, 2, 3$ are k-group schemes.

A quick calculation on the Hopf algebra shows that the affine algebra for every finite flat group scheme G of height one over k is of the form $k[X_1, \cdots, X_d]/(X_1^p, \cdots, X_d^p)$, with $d = \dim_k \mathrm{Lie}(G/k)$. In particular, $\mathrm{ord}(G) = p^d$. As $\mathrm{ord}(G_2) = \mathrm{ord}(G_1) \cdot \mathrm{ord}(G_3)$, $\dim_k \mathrm{Lie}(G_2/k) = \dim_k \mathrm{Lie}(G_1/k) + \dim_k \mathrm{Lie}(G_3/k)$. And the right exactness follows.

Let
$$0 \to L_1 \to L_2 \to L_3 \to 0$$
be a short exact sequence of locally free restricted Lie algebras over S. We want to show $G_p(\cdot)$ is exact.

We notice that it suffices to show the induced complex
$$0 \to G_p(L_1) \to G_p(L_2)$$
is exact. Then one can extend the sequnce to
$$0 \to G_p(L_1) \to G_p(L_2) \to G \to 0$$
by Lemma 2.2. In particular, G is finite flat with height at most one obviously. Since $\mathrm{Lie}(\cdot/S)$ preserves short exact sequences, $\mathrm{Lie}(G/S) \cong L_2/L_1 \cong L_3$. By Theorem 3.7, $G \cong G_p(L_3)$ and we are finished.

Now we show that $G_p(L_1) \to G_p(L_2)$ is a closed embedding. By definition, $G_p(L_i) = \underline{\mathrm{Spec}}_{\mathcal{O}_S}(U_p(L_i)^D)$, $i = 1, 2$. So, it suffices to show the natural map $U_p(L_2)^D \to U_p(L_1)^D$ is a surjective $\mathcal{O}_S$-algebra map.

It suffices to do this locally. Suppose $S = \mathrm{Spec}(A)$ and
$$0 \to L_1 \to L_2 \to L_3 \to 0$$

is an exact sequence of finite free A-modules. Let $x_1, ..., x_n$ be a basis of L_1. Since L_3 is also finite free, a basis $y_1, ..., y_m$ of L_3 can be lifted to L_2 so that $x_1, ...x_n, y_1, ..., y_m$ gives a basis of L_2.

By the restricted PBW theorem ([DGA11, Proposition 5.3.3]), $U_p(L_1)$ has basis $x_1^{a_1} \cdots x_n^{a_n}$ for $0 \le a_i < p$ and $U_p(L_2)$ has basis $x_1^{b_1} \cdots x_n^{b_n} y_1^{c_1} \cdots y_m^{a_m}, 0 \le b_j, c_j < p$.

In particular, the morphism between finite A-algebras $i : U_p(L_1) \to U_p(L_2)$ is injective. And the dual $i^D : U_p(L_2)^D \to U_p(L_1)^D$ is surjective: let $\xi_{\underline{a}}$ be the dual basis of $x^{\underline{a}}$ in $U_p(L_1)^D$, where $a = (a_1, ..., a_n)$ and let $\eta_{(\underline{b},\underline{c})}$ be the dual basis of $x^{\underline{b}} y^{\underline{c}}$ in $U_p(L_2)^D$ similarly. Then $i^D(\eta_{(\underline{a}',\underline{0})})(x^{\underline{a}}) = \eta_{(\underline{a}',\underline{0})}(i(x^{\underline{a}})) = \eta_{(\underline{a}',\underline{0})}(x^{\underline{a}} y^{\underline{0}}) = \delta_{\underline{a},\underline{a}'}$. In particular, $i^D(\eta_{(\underline{a},0)}) = \xi_{\underline{a}}$ and i^D is hence surjective. □

Remark 4.2. Notice that one need to let L_3 be locally free, otherwise the basis of L_1 may not extend to L_2. For example, let $A = \mathbb{F}_p[t]$, $L_1 = t\mathbb{F}_p[t]$ and $L_2 = \mathbb{F}_p[t]$ with trivial p-mapping. Let x denote the generator of L_2 over A. Then $y = tx$ is the generator of L_1 over A.

The inclusion morphism $i : L_1 \to L_2$ induces an injective morphism $i : U_p(L_1) \to U_p(L_2)$ sending $a_0 + a_1 y + \cdots a_{p-1} y^{p-1}$ to $a_0 + a_1 tx + \cdots a_{p-1} t^{p-1} x^{p-1}$.

However, $i^D : U_p(L_2)^D \to U_p(L_1)^D$ fails to be surjective: let η_i, ξ_j be the dual of x^i, y^j respectively. Then $i^D(\eta)(y) = \eta(ty) = t\xi(y)$. In particular, $i^D(\eta) = t\xi$ and i^D fails to be surjective.

Corollary 4.3. Lie$(\cdot/S)$ preserves injections and surjections.

Proof. Lie$(\cdot/S)$ preserves injections and surjections as one can extend the morphisms to short exact sequences by Lemma 2.2. □

Remark 4.4. We note that $G_p(\cdot)$ does not preserve injections in general: let $\phi : V \to W$ be an injective morphism of locally free restricted Lie algebras. Since the cokernel of ϕ may fail to be locally free, ϕ might not correspond to a closed immersion of group schemes by Lemma 2.2.

Here is an example: let E be a supersingular elliptic curve over $\overline{\mathbb{F}}_p$. Let $Y_0 \coloneqq E \times \mathbb{P}^1$ be a constant abelian scheme over $\mathbb{P}^1$. Consider the subsheaf $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{O}_{\mathbb{P}^1} \cong \mathrm{Lie}(Y_0/\mathbb{P}^1)$ given by multiplication by a nonzero element $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. This embedding can be considered as a map of restricted Lie algebras, as the p-mapping on $\mathrm{Lie}(Y_0/\mathbb{P}^1)$ is zero.

Suppose this corresponds to a closed group scheme H of $Y_0[F]$. Then the fppf quotient Y_0/H is an abelian scheme with Lie algebra satisfying the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathrm{Lie}(Y/\mathbb{P}^1) \to \mathcal{O}_{\mathbb{P}^1}(-p) \to 0,$$

which is a contradiction to the local freeness of $\mathrm{Lie}(Y/\mathbb{P}^1)$.

We explain why it fails categorically: if ϕ is an injection, it is a monomorphism in the category of locally free sheaves and hence a monomorphism in the category of locally free restricted Lie algebras. It then corresponds to a monomorphism $G_p(\phi) : G_p(V) \to G_p(W)$ in the category $\mathrm{FFGp}_S^{F\text{-tor}}$. But it may fail to be a monomorphism in the larger category $\mathrm{FPPF}(S)$ (or Sch/S). So, the induced morphism $G_p(\phi)$ may not be a closed immersion by Lemma 2.2. In particular, we have the following criterion.

Corollary 4.5. (1) $G_p(\cdot)$ preserves injections if and only if the cokernel is locally free.
(2) $G_p(\cdot)$ preserves surjections.

Proof. (1). Let $\phi : V \to W$ be an injection of locally free restricted Lie algebras. If $G_p(\phi) : G_p(V) \to G_p(W)$ is an injection, Lemma 2.2 shows that one can extend $G_p(\phi)$ to a short exact sequence and the cokernel of ϕ is hence locally free as being Lie algebra of $G_p(W)/G_p(V)$. Conversely, if the cokernel of ϕ is locally free, one has the following commutative diagram as F_S^* is exact for exact sequences of locally free sheaves

$$\begin{tikzcd}
0 \arrow[r] & F_S^*V \arrow[r] \arrow[d, "\varphi_V"] & F_S^*W \arrow[r] \arrow[d, "\varphi_W"] \arrow[dr] & F_S^*(W/V) \cong F^*W/F^*V \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & V \arrow[r] & W \arrow[r] & W/V \arrow[r] & 0
\end{tikzcd}$$

In particular, this shows that ϕ can be extended to a short exact sequence of locally free restricted Lie algebras, and the conclusion follows from Proposition 4.1.

(2). Let $\phi : V \to W$ be a surjection of locally free restricted Lie algebras. Then $\mathrm{Ker}(\phi)$ is locally free. As F_S^* is exact for exact sequences of locally free sheaves, one has the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_S^*\mathrm{Ker}(\phi) \arrow[r] \arrow[d, dashed] \arrow[dr, "\varphi_V"] & F_S^*V \arrow[r, "F_S^*\phi"] \arrow[d, "\varphi_V"] & F_S^*W \arrow[r] \arrow[d, "\varphi_W"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\phi) \arrow[r] & V \arrow[r, "\phi"] & W \arrow[r] & 0
\end{tikzcd}$$

It follows that

$$0 \to \mathrm{Ker}(\phi) \to V \to W \to 0$$

is a short exact sequence in the category of locally free restricted Lie algebras. And the conclusion follows from Proposition 4.1. □

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