A DISCUSSION ON MAXIMAL VARIATIONS OF ABELIAN SCHEMES

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1. Introduction

This survey concerns about such a question:

Question 1.1. Let $\mathcal{X}/C$ be an abelian scheme over a smooth proper curve S over some field k. Under what conditions on the Hodge bundle of $\mathcal{X}/C$ does A contain a constant abelian subscheme over C? [1]

For our convenience, we distinguish several definitions of "maximal invariant". Necessary definitions will be introduced later.

Definition 1.2. Let C be a smooth proper curve over a perfect field k (can be any characteristic). Let $\mathcal{X}/C$ be an abelian scheme of relative dimension g. Let $K \coloneqq K(C)$ be the function field so that K/k is a regular extension. Let $X \coloneqq \mathcal{X}_K$ be the generic fibre.

(1) We say that $\mathcal{X}/C$ has geometrically maximal variation (GMV) if there does not exists an abelian variety Y/k such that $Y \times C \hookrightarrow \mathcal{X}$ is an abelian subscheme of constant type.
(2) We say that $\mathcal{X}/C$ has monodromically maximal variation (MMV) if $\mathrm{Tr}_{K/k}X = 0$.
(3) We say that $\mathcal{X}/C$ has infinitesimally maximal variation (IMV) if $H^0(C, \mathrm{Lie}(\mathcal{X}/C)) = 0$.

Remark 1.3. The terminology monodromically maximal variation is motivated by the fact that, although the kernel of the K/k-trace may be a nontrivial connected finite group scheme, this has no effect on the ℓ-adic Tate module.

Remark 1.4. Some authors define maximal variation in an "absolute" sense: an abelian scheme $\mathcal{X}/C$ is said to have GMV (resp. MMV, IMV) if the base change $\mathcal{X}_D/D$ has GMV (resp. MMV, IMV) for every finite étale cover $D \to C$. We note that all three notions introduced above are not invariant under finite étale base change. Therefore, throughout this survey, we keep the relative definitions without adding the adjective "absolute".

However, when one only considers coarse properties of the Hodge bundle, it is often harmless, and sometimes convenient, to pass to a finite étale cover. For instance, if the Hodge bundle has degree zero, then the abelian scheme $\mathcal{X}/C$ becomes isotrivial over C; namely, there exists a finite étale cover $D \to C$ such that $\mathcal{X}_D \simeq Y \times D$ for some abelian variety Y over the base field (see [Yua21, Theorem 2.6]).

We summarize the results of this survey in the following graph.

$$
\begin{tikzcd}
\mathrm{IMV} \arrow[r, Rightarrow, "\text{Proposition 3.1}"] & \mathrm{MMV} \arrow[l, Rightarrow, bend right=40, "\mathrm{char}(k)=0\text{,Proposition 4.1}"'] \arrow[l, Rightarrow, bend left=40, "\times" marking, "\mathrm{char}(k)=p\text{,Proposition 5.2}"] \arrow[r, Rightarrow, "\text{Proposition 3.1}"] & \mathrm{GMV} \arrow[l, Rightarrow, bend right=40, "\mathrm{char}(k)=0\text{,Proposition 3.2}"'] \arrow[l, Rightarrow, bend left=40, "\times" marking, "\mathrm{char}(k)=p\text{,Proposition 5.1}"]
\end{tikzcd}
$$

Date: July 31, 2026.

[1] The question was motivated by Professor Rössler's question and Professor Starr's reply on mathoverflow: `https://mathoverflow.net/questions/166193/ampleness-of-hodge-bundles-over-complex-curves`.

2. Preliminaries

2.1. **Chow's K/k-trace.** To begin with, we first give a review on the theory of Chow's trace for abelian varieties.

Let K/k be a primary extension of fields, i.e., the algebraic closure of k in K is purely inseparable over k. If moreover, k is algebraically closed in K and K/k is separable, then we also call such K/k a regular extension. For example, if k is perfect and K is the function field of a smooth and geometrically connected k-scheme, K/k is a regular extension.

Definition 2.1 (Chow's K/k-trace)**.** Let K/k be a primary extension and let X be an abelian variety over K. The K/k-trace is a final object in the category of pairs (B, ϕ), where B is a k-abelian variety and $\phi : B_K \to X$ is a K-homomorphism of abelian varieties.

We refer the reader to read Brain Conrad's article [Con06] for more details about the K/k-trace. We recall the following properties that we will use later.

Theorem 2.2 ([Con06], Section 6)**.** Let K/k be a primary extension and let X be an abelian variety over K.

(1) The K/k-trace
$$\tau = \tau_{X,K/k} : (\mathrm{Tr}_{K/k}X)_K \to X$$
always exists. Moreover, $\mathrm{Ker}(\tau)$ is a finite flat group scheme over K with connected Cartier dual.
(2) If K/k is furthermore a regular extension, then $\mathrm{Ker}(\tau)$ is connected.
(3) There exists a unique abelian subvariety $X' \subseteq X$ such that $\mathrm{Tr}_{K/k}(X/X') = 0$ and $\tau_{X',K/k}$: $\mathrm{Tr}_{K/k}(X')_K \to X'$ is an isogeny.
(4) Let k'/k be an arbitrary extension. Suppose that either k'/k is separable or K/k is regular. Then the base change $((\mathrm{Tr}_{K/k}X) \otimes_k k', \tau \otimes_K Kk')$ gives the Kk'/k'-trace of $X_{Kk'}$.
(5) If K'/K is a primary extension, then $(\mathrm{Tr}_{K/k}X, \tau_{K'})$ is the K'/k-trace of $X_{K'}$.-

In particular, we see that if K/k is a regular extension, the K/k-trace $\tau : (\mathrm{Tr}_{K/k}X)_K \to X$ gives a K-isogeny from $(\mathrm{Tr}_{K/k}X)_K$ onto its image. It is not always the case that τ is an isogeny itself.

On the other hand, $\mathrm{Tr}_{K/k}X$ is the natural object for studying the isotriviality of an abelian scheme $\phi : \mathcal{X} \to C$ over a smooth proper k-curve C. Let k be a perfect field and let $K = K(C)$ be the function field, so that K/k is a regular extension. Let $X = \mathcal{X}_K$ be the generic fibre of $\mathcal{X}$.

By the theory of Néron models, the trace map $\tau : (\mathrm{Tr}_{K/k}X)_K \to X$ extends a homomorphism of abelian schemes
$$\widetilde{\tau} : \mathrm{Tr}_{K/k}X \times_{\mathrm{Spec}(k)} C \to \mathcal{X}$$
over C. By the universal property of the trace, any homomorphism from a constant abelian scheme $B \times C \to \mathcal{X}$ factors through $\widetilde{\tau}$. So, the existence of a nontrivial homomorphism from a constant abelian scheme to $\mathcal{X}$ forces the K/k-trace to be nontrivial.

2.2. **Hodge bundle and Lie algebra.** Let S be a locally Noetherian scheme over a field k. Let $f : \mathcal{X} \to S$ be an abelian scheme of relative dimension g. Then the first de Rham cohomology $\mathbb{H}^1_{dR}(\mathcal{X}/S)$ admits a Hodge filtration
$$\mathrm{Fil}_h^\bullet : 0 \subset \mathrm{Fil}^1 \subset \mathrm{Fil}^0,$$
with $\mathrm{Fil}^1 := f_*\Omega^1_{\mathcal{X}/S}$ and $\mathrm{Fil}^0 = \mathbb{H}^1_{dR}(\mathcal{X}/S)$. The Hodge bundle of $\mathcal{X}/S$ is then defined to be the sheaf $f_*\Omega^1_{\mathcal{X}/S}$, which is known to be locally free of rank g. One also has the graded piece
$$\mathrm{Gr}^0 := \mathrm{Fil}^0/\mathrm{Fil}^1 \cong R^1 f_*\mathcal{O}_{\mathcal{X}/S},$$
which is also locally free of rank g.

On the other hand, one can define the (relative) Lie algebra

$$\mathrm{Lie}(G/S) \coloneqq s^*(\Omega^1_{G/S})^\vee$$

for any group scheme $f : G \to S$ with unit section $s : S \to G$. In other words, it is the sheaf of invariant vector fields over S. Notice that for abelian schemes $\mathcal{X}/S$, one has a natural identification

$$f_*\Omega_{\mathcal{X}/S} \cong s^*\Omega^1_{\mathcal{X}/S},$$

as any differential form for abelian varieties is invariant automatically. So, one has a natural identification

$$\mathrm{Lie}(\mathcal{X}/S) \cong (f_*\Omega_{\mathcal{X}/S})^\vee.$$

It is also known that one has a canonical isomorphism of fppf-sheaves of $\mathcal{O}_S$-modules

$$R^1 f_*\mathcal{O}_{\mathcal{X}/S} \xrightarrow{\sim} \mathrm{Lie}(\mathcal{X}^\vee/S).$$

In particular, if $\mathcal{X}/S$ is a principally polarized abelian scheme, one has a canonical isomorphism of locally free $\mathcal{O}_S$-modules

$$R^1 f_*\mathcal{O}_{\mathcal{X}/S} \cong \mathrm{Lie}(\mathcal{X}/S),$$

and hence a short exact sequence

$$0 \to f_*\Omega^1_{\mathcal{X}/S} \to \mathbb{H}^1_{dR}(\mathcal{X}/S) \to \mathrm{Lie}(\mathcal{X}/S) \to 0.$$

Remark 2.3. If $\mathrm{char}(S) = 0$, then one always has $\mathrm{Lie}(\mathcal{X}/S) \cong \mathrm{Lie}(\mathcal{X}^\vee/S)$ without conditions on polarizations, as $\mathrm{Lie}(\cdot/S)$ is a left exact functor on group schemes and the finite étale group schemes have trivial Lie algebras.

If $\mathrm{char}(S) > 0$, there further exists an equivalence of categories between the category of finite flat group schemes over S of height at most one and the category of finite locally free restricted Lie algebras over S, given by the functor $G/S \to \mathrm{Lie}(G/S)$. For a detailed discussion of finite flat group schemes and restricted Lie algebras, we refer the reader to the author's other notes `https://haochencheng1.github.io/restricted_lie_algebra.pdf`. We only need the following theorem:

Theorem 2.4. *Let S be a locally Noetherian scheme of characteristic $p > 0$. Then there is an equivalence of categories*

$$\mathrm{Lie}(\cdot/S) : \mathrm{FFGp}_S^{F\text{-tor}} \longrightarrow p\text{-}\mathrm{Lie}_{\mathcal{O}_S}, G \mapsto \mathrm{Lie}(G/S)$$

between the category of commutative fintie flat group schemes of height at most one and the category of locally free commutative restricted Lie algebra over S.

Furthermore, the category p-$\mathrm{Lie}_{\mathcal{O}_S}$ can be characterized the category consisting of objects as pairs

$$\{(\mathcal{L}, \phi); \mathcal{L} \in \mathrm{Vect}_S, \phi \in \mathrm{Hom}_{\mathcal{O}_S}(F_S^*\mathcal{L}, \mathcal{L})\}$$

and morphisms as compatible sheaf morphisms.

2.3. **Positivity of Hodge bundles.** Let k be a field. Let C be a smooth proper curve over k and let $\mathcal{E} \in \mathrm{Vect}_C$ be a rank g vector bundle over C.

Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{E}$$

be the Harder-Narasimhan filtration of $\mathcal{E}$. It is the unique filtration of subbundles of $\mathcal{E}$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable and satisfies

$$\mu(\mathcal{F}_1/\mathcal{F}_0) > \cdots > \mu(\mathcal{F}_m/\mathcal{F}_{m-1}).$$

We will write $\mu_{\min}(\mathcal{E}) \coloneqq \mu(\mathcal{F}_m/\mathcal{F}_{m-1})$ and $\mu_{\max}(\mathcal{E}) \coloneqq \mu(\mathcal{F}_1)$.

If $\mathrm{char}(k) = 0$, it is well known that $\mathcal{E}$ is ample (resp. nef) if and only if $\mu_{\min}(\mathcal{E}) > 0$ (resp. $\mu_{\min}(\mathcal{E}) \geq 0$).

If $\mathrm{char}(k) = p > 0$, one has to involve the (absolute) Frobenius F_C. The reason is that the pullback of an ample (resp. nef) bundle along a finite morphism (for example, F_C) is still ample (resp. nef), but the Frobenius pullback of a semistable bundle may fail to be semistable (see [Gie73]).

We call a vector bundle $\mathcal{E}$ is strongly semistable if $(F_C^n)^*\mathcal{E}$ is semistable for all $n \geq 0$. As shown in [Gie73] again, an arbitrary vector bundle does not have a Harder-Narasimhan filtration by strongly semistable vector bundles in general. However, $(F_C^m)^*$ does have for some $m \gg 0$ (see [Lan04, Theorem 2.7]).

In particular, the following numbers are well-defined:

$$\bar{\mu}_{\min}(\mathcal{E}) = \varinjlim_{n\to\infty} \frac{\mu_{\min}((F_C^n)^*\mathcal{E})}{p^n},$$

$$\bar{\mu}_{\max}(\mathcal{E}) = \varinjlim_{n\to\infty} \frac{\mu_{\max}((F_C^n)^*\mathcal{E})}{p^n}.$$

It is obvious that $\bar{\mu}_{\max}(\mathcal{E}) \geq \mu_{\max}(\mathcal{E})$ and $\bar{\mu}_{\min}(\mathcal{E}) \leq \mu_{\min}(\mathcal{E})$.

Moreover, if $\mathrm{char}(k) = p > 0$, then $\mathcal{E}$ is ample (resp. nef) if and only if $\bar{\mu}_{\min}(\mathcal{E}) > 0$ (resp. $\bar{\mu}_{\min}(\mathcal{E}) \geq 0$) (see [Bar71]).

Now, let $\mathcal{E} := f_*\Omega_{\mathcal{X}/C}$ be the Hodge bundle of a family of abelian varieties $f : \mathcal{X} \to C$ over a smooth proper curve C over k.

If $\mathrm{char}(k) = 0$, $f_*\Omega_{\mathcal{X}/C}$ is nef (see [Bos04]) and hence $\mu_{\max}(\mathrm{Lie}(\mathcal{X}/C)) \leq 0$.

If $\mathrm{char}(k) = p > 0$, it is no longer true that $f_*\Omega_{\mathcal{X}/C}$ is nef (see [Che26]).

3. Maximal variations of abelian schemes

Let k be a field. Let C be a smooth proper curve over k. Let $\mathcal{X}/C$ be a principally polarzied abelian scheme of dimension g. Let $K = K(C)$ be the function field and $X := \mathcal{X}_K$ the generic fibre. We fix the notation $B := \mathrm{Tr}_{K/k}X$ for the K/k-trace of X.

Proposition 3.1. For any field k, one has the relation

$$\mathrm{IMV} \Longrightarrow \mathrm{MMV} \Longrightarrow \mathrm{GMV}.$$

Proof. We first show the first arrow IMV $\Longrightarrow$ MMV. Suppose $B := \mathrm{Tr}_{K/k}X \neq 0$, then there exists a morphism of abelian schemes

$$\tau : B \times C \to \mathcal{X}$$

by the theory of Néron models.

We want to show $H^0(C, \mathrm{Lie}(\mathcal{X}/C)) \neq 0$. Since $\mathrm{Lie}(B \times C/C) \cong \mathcal{O}_C^{\oplus g}$, it suffices to show the induced map

$$\mathrm{Lie}(\tau) : \mathrm{Lie}(B \times C/C) \to \mathrm{Lie}(\mathcal{X}/C)$$

is nonzero.

If $\mathrm{char}(k) = 0$, τ_K is injective by Theorem 2.2 (2). Consider the following exact sequence of group schemes

$$0 \to \mathrm{Ker}(\tau) \to B \times C \to \mathcal{X}.$$

Since pullback is exact in the category of FPPF sheaves over C, one has $\mathrm{Ker}(\tau)_K = \mathrm{Ker}(\tau_K) = 0$.

On the other hand, $\mathrm{Lie}(\cdot/C)$ is left exact in the category of group schemes over C So, one has an exact sequence of Lie algebras

$$0 \to \mathrm{Lie}(\mathrm{Ker}(\tau)/C) \to \mathrm{Lie}(B \times C/C) \xrightarrow{\mathrm{Lie}(\tau)} \mathrm{Lie}(\mathcal{X}/C).$$

If $\mathrm{Lie}(\tau) = 0$, then $\mathrm{Lie}(\mathrm{Ker}(\tau)/C) \cong \mathrm{Lie}(B \times C/C) \cong \mathcal{O}_C^{\oplus g}$. In particular, $\mathrm{Lie}(\mathrm{Ker}(\tau)_K/K) \cong K^{\oplus g}$ is nonzero, which is a contradiction.

If $\mathrm{char}(k) = p > 0$, suppose $\mathrm{Lie}(\tau) = 0$. Then $\tau : B[F] \times C \to \mathcal{X}[F]$ is a trivial map by Theorem 2.4. In particular, $B[F] \times C \subseteq \mathrm{Ker}(\tau)$ and $\tau = \tau' \circ \mathrm{Frob}_{B\times C/C}$ factors through the

Frobenius morphism. Since $(B \times C/B[F] \times C) \cong (B/B[F]) \times C \cong B^{(p/k)} \times C$ is still constant type, this gives a contradiction to the universal property of K/k-trace.

The second arrow MMV $\Longrightarrow$ GMV is just the universal property of K/k-trace by definition. $\square$

Proposition 3.2. If char(k)=0, then GMV implies MMV.

Proof. Suppose $\mathrm{char}(k) = 0$ and $\mathrm{Tr}_{K/k}X \neq 0$. By Theorem 2.2 (2), $\tau_K : \mathrm{Tr}_{K/k}X \to X$ is an abelian subvariety. By the theory of Néron models, τ_K extends to a morphism of abelian schemes

$$\tau : \mathrm{Tr}_{K/k}X \times_k C \to \mathcal{X}.$$

One then can finish the proof by using the fact that if char(k)=0, then the map between Néron models is a closed immersion if the generic fibre is (see [BLR90, Section 7.5]). $\square$

Remark 3.3. There are two obstructions for Proposition 3.2 in positive characteristic: the kernel of the trace morphism may be non-trivial, and the closed immersion of generic fibre may not extend to a closed immersion of Néron models in positivie charactersitc.

We refer readers to [BLR90, Chapter 7] and [ACW24] for more details, where the question was carefully studied.

We just note that even for morphism of abelian schemes $f : \mathcal{X} \to \mathcal{Y}$ over a smooth proper curve C over a field k of characteristic $p > 0$, $\mathrm{Ker}(f)$ can non-zero even if the Kernel of its generic fibre is zero (remember that since $\mathcal{X}$ is proper over C, f is closed immersion if and only if $\ker(f) = 0$ by [Sta25, Lemma 04XV]).

In particular, the induced morphism on Néorn models may fail to be flat.

A counter-example was given in [ACW24, Example 4.13]. We give another counter-example here: Let $\phi : C \to \mathcal{A}_{g,n,k}$ be a morphism from a smooth proper curve to the moduli space of abelian varieties of dimension g and level-n-strucutre with $n \geq 3$ over $k = \overline{\mathbb{F}}_p$. Suppose $\phi(C)$ contains both an ordinary and a superspecial point. Such example can be constructed like this: consider the Satake compactification $\mathcal{A}^*_{g,n,k}$ of $\mathcal{A}_{g,n,k}$.

It is a projective variety over k with $\mathrm{codim}(\mathcal{A}^*_{g,n,k} \setminus \mathcal{A}_{g,n,k}) = g$. Let $i : \mathcal{A}^*_{g,n,k} \to \mathbb{P}^{g(g+1)/2}_k$ be a closed embedding. Choose $x \in \mathcal{A}^{\mathrm{ord}}_{g,n,k}(k)$ and y a superspecial point. Then there exists a set of general hyperplanes $H_1, \cdots, H_{g(g+1)/2-1}$ in $\mathbb{P}^{g(g+1)/2}_k$ such that $H_1 \cap \cdots H_{g(g+1)/2-1} \cap \mathcal{A}^*_{g,n,k}$ is a projective curve contains x, y and avoids $\mathcal{A}^*_{g,n,k} \setminus \mathcal{A}_{g,n,k}$ by Bertini theorem. Taking normalization if necessary, we obtain a morphism from a smooth proper curve $\phi : C \to \mathcal{A}_{g,n,k}$ that contains x, y in $\phi(C)$.

Let $f : \mathcal{X} \to C$ be the universal abelian scheme. Let $\eta = \mathrm{Spec}(K(C)) \in C$ be the ordinary point so that $X := \mathcal{X}_K$ is ordinary. Let $F_{\mathcal{X}/C} : \mathcal{X} \to \mathcal{X}^{(p/C)}$ and $V_{\mathcal{X}^{(p)}/C} : \mathcal{X}^{(p/C)} \to \mathcal{X}$ be the Frobenius and Vershiebung morphism over C respectively.

Consider the morphism of abelian schemes over C:

$$\Phi = (F_{\mathcal{X}^{(p/C)}/C}, V_{\mathcal{X}/C}) : \mathcal{X}^{(p/C)} \to \mathcal{X}^{(p^2/C)} \times_C \mathcal{X}.$$

For any point $s \in C$, $\mathrm{Ker}(\Phi_s) \cong \mathrm{Ker}(V_{\mathcal{X}_s/k(s)}) \cap \mathrm{Ker}(F_{\mathcal{X}_s^{(p/k(s))}/k(s)})$. Let $s = \eta$. Then $\mathcal{X}_s$ is ordinary. So, $\mathrm{Ker}(V_{\mathcal{X}_s/k(s)})$ is étale and $\mathrm{Ker}(F_{\mathcal{X}_s^{(p/k(s))}/k(s)})$ is connected. In particular, $\mathrm{Ker}(\Phi_s) = 0$.

Let $s = y'$ be one pre-image of y. Then $\mathcal{X}_s$ is superspecial. In particular, $\mathcal{X}_s \cong E_1 \times \cdots \times E_g$ is isomorphic to a product of supersingular elliptic curves. In particuar, $\alpha_p^{\oplus g}$ is a closed subgroup of $\mathcal{X}_s$. And it is obvious that $\mathrm{Ker}(\Phi_s) \neq 0$.

So, Φ is generically zero but fail to be zero itself.

4. MMV implies IMV in $\mathbb{C}$

The main result of this section is the following:

Proposition 4.1. If $\mathrm{char}(k) = 0$, then MMV implies IMV.

We notice that it suffices to show Proposition 4.1 when $k = \mathbb{C}$: let $\mathcal{X}/C$ be an abelian scheme over a smooth proper curve defined over k, with $\mathrm{char}(k) = 0$. Then there exists a finitely generated field $\mathbb{Q} \subseteq k_0 \subseteq k$ such that $\mathcal{X}/C$ can be defined over k_0. It is known that any finitely generated field of characteristic 0 can be embedded into $\mathbb{C}$. Then Proposition 4.1 follows from the complex field case as both of the conditions, the existence of trace and the existence of nontrivial global sections of vector bundles, are invariant under separable field extension (see Theorem 2.2 (4)).

We fix a complex smooth proper curve C and a family of abelian varieties $f : \mathcal{X} \to C$. We write $\mathcal{E} := f_*\Omega_{\mathcal{X}/C}$ and thus $\mathrm{Lie}(\mathcal{X}/C) \cong \mathcal{E}^\vee$ by our discussion before.

We write $\theta : \mathcal{E} \to \mathcal{E}^\vee \otimes \Omega^1_{C/\mathbb{C}}$ for the Higgs field (Kodaira-Spencer map) induced by the Gauss-Manin connection. Write $(\mathcal{E} \oplus \mathcal{E}^\vee, \theta)$ for the assoicated Higgs bundle.

Proposition 4.2 (Simpson correspondence, [Sim92])**.** Over C, there is an equivalence of categories between the category of semisimple flat bundles on C and the category of polystable Higgs bundles of slope zero.

Theorem 4.3 (Fujita decomposition)**.** Let $f : \mathcal{X} \to C$ be a family of abelian varieties, then the Hodge bundle has a decomposition

$$f_*\Omega_{\mathcal{X}/C} \cong \mathcal{A} \oplus \mathcal{U},$$

with $\mathcal{A}$ ample and $\mathcal{U}$ flat for the Gauss-Manin connection.

In particular, the Higgs bundle has a decomposition

$$(\mathcal{E} \oplus \mathcal{E}^\vee, \theta) \cong (\mathcal{A} \oplus \mathcal{A}^\vee, \theta) \oplus (\mathcal{U}, 0) \oplus (\mathcal{U}^\vee, 0).$$

Suppose $H^0(C, \mathrm{Lie}(\mathcal{X}/C)) \neq 0$. Then there exists a free subbundle $\mathcal{O}_C^{\oplus m} \subseteq \mathcal{U}^\vee$. But since $(\mathcal{E} \oplus \mathcal{E}^\vee, 0)$ is Higgs polystable by Proposition 4.2, one must have $(\mathcal{O}_C^{\oplus m}, 0) \subseteq (\mathcal{U}^\vee, 0)$ is a direct summand. In particular, $\mathcal{U} \cong \mathcal{O}_C^{\oplus m} \oplus \mathcal{F}$ for some semistable bundle $\mathcal{F}$ of degree zero and $H^0(C, \mathcal{F}) = 0$.

By Proposition 4.2 and Riemann-Hilbert, the free Higgs subbundle $(\mathcal{O}_C^{\oplus 2m}, 0) \subseteq (\mathcal{E} \oplus \mathcal{E}^\vee, \theta)$ induces a trivial $\mathbb{C}$-local system $\underline{\mathbb{C}}^{2m} \subset R^1 f_*^{an}\underline{\mathbb{C}}$. In particular, $H^0(X, R^1 f_*^{an}\underline{\mathbb{C}}) \neq 0$. And also $H^0(X, R^1 f_*^{an}\underline{\mathbb{Q}}) \neq 0$.

Theorem 4.4. If $H^0(C, R^1 f_*^{an}\underline{\mathbb{Q}}) = 0$ if and only if $\mathcal{X}/C$ has MMV.

Proof. Suppose $H^0(C, R^1 f_*^{an}\underline{\mathbb{Q}}) \neq 0$. In particular, the fixed part $\mathbb{V}_0 := \underline{\mathbb{Z}}^{\oplus h^0(R^1 f_*^{an}\underline{\mathbb{Q}})} \subset \mathbb{V} := R^1 f_*^{an}\underline{\mathbb{Q}}$ is a nontrivial local subsystem.

By the theorem of fixed part ([Del71, Corollary 4.1.2]), $\mathbb{V}_0$ becomes a constant $\mathbb{Q}$-sub-VHS of $\mathbb{V}$. Let $(\mathbb{V}_0)_\mathbb{Z}$ be an integral lattice of $\mathbb{V}_0$. This becomes a nontrivial constant $\mathbb{Z}$-sub-VHS of $R^1 f_*^{an}\underline{\mathbb{Z}}$. $(\mathbb{V}_0)_\mathbb{Z}$ inherites the polarization on $R^1 f_*^{an}\underline{\mathbb{Z}}$ by definition. So, $(\mathbb{V}_0)_\mathbb{Z}$ is a nontrivial constant $\mathbb{Z}$-sub-PVHS of $R^1 f_*^{an}\underline{\mathbb{Z}}$. In particular, $\mathcal{X}$ has a constant abelian subscheme over C and $\mathcal{X}/C$ does not have MMV.

Conversely, suppose $\mathcal{X}/C$ does not have MMV. So, there exists a nontrivial finite morphism $B \times C \to \mathcal{X}$ for some complex abelian variety B of dimension g_0. It induces a nontrivial map of $\mathbb{Q}$-local system $\underline{\mathbb{Q}}^{\oplus 2g_0} \to R^1 f_*^{an}\underline{\mathbb{Q}}$. In particular, $H^0(C, R^1 f_*^{an}\underline{\mathbb{Q}}) \neq 0$.

□

Proof of Proposition 4.1. The proposition then follows from the above discussion. □

5. The failness in characteristic p

We fix an algebraically closed field $k = \overline{\mathbb{F}}_p$ of characteristic $p > 0$. In this section, we aim to show that the equivalence among the three maximal variations in characteristic zero will fail in characteristic p.

5.1. GMV does not imply MMV in characteristic p.

Proposition 5.1. GMV does not imply MMV in characteristic p.

The examplle is again given by the Moret-Bailly family. We recall it now. In [Spz+81], Moret-Bailly shows that there exists a superspecial abelian surface $X_0 = E_1 \times E_2$ with an ample bundle $\mathcal{L}$ such that $K(\mathcal{L}) = \alpha_p \times \alpha_p$.

Let $\mathcal{X}_0 := X_0 \times \mathbb{P}^1_k$ be a constant abelian scheme over $\mathbb{P}^1_k$. It has restricted Lie algebra $\mathrm{Lie}(\mathcal{X}_0/\mathbb{P}^1_k) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ with zero p-mapping. So, one can consider the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ with an exact sequence of vector bundles

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Associated to $\mathcal{O}_{\mathbb{P}^1}(-1)$ a trivial morphism $\mathcal{O}_{\mathbb{P}^1}(-p) \to \mathcal{O}_{\mathbb{P}^1}(-1)$, the subbundle $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ corresponds to a finite flat closed group subscheme H of $K(\mathcal{L}) \times \mathbb{P}^1$ that is locally isomorphic to α_p.

Let $\mathcal{X} := \mathcal{X}_0/H$ be the fppf quotient. Then $\mathcal{X}$ becomes a supersingular abelian scheme over $\mathbb{P}^1$ and has principal polarization given by $\mathcal{L}$. Moreover, one has an identification $\mathrm{Lie}(\mathcal{X}/\mathbb{P}^1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-p)$.

Let $K = K(\mathbb{P}^1)$ and $X := \mathcal{X}_K$. Then it is clear that $\mathrm{Tr}_{K/k} X \neq 0$, as there is an isogeny

$$\tau_K : (X_0)_K \to X.$$

On the other hand, $\mathcal{X}$ does not have an abelian subscheme $B \times \mathbb{P}^1$ for some abelian variety B/k. The intuitive reason is that $\mathbb{P}^1_k$ gives an irreducible component of the moduli space of supersingular abelian surfaces $\mathcal{S}_2$. So, it cannot have a constant subfamily.

Explicitly, if $\mathcal{X}$ has an abelian subscheme $B \times \mathbb{P}^1$. Then the fppf quotient $\mathcal{X}' := \mathcal{X}/(B \times \mathbb{P}^1)$ is still an abelian scheme. Then the exact sequence of fppf sheaves

$$0 \to B \times \mathbb{P}^1 \to \mathcal{X} \to \mathcal{X}' \to 0$$

induces an exact sequence of fppf sheaves

$$0 \to B[F] \times \mathbb{P}^1 \to \mathcal{X}[F] \to \mathcal{X}'[F] \to 0$$

by Snake lemma, and hence induces an exact sequence of locally free sheaves over $\mathbb{P}^1$

$$0 \to \mathrm{Lie}(B \times \mathbb{P}^1/k) \to \mathrm{Lie}(\mathcal{X}/\mathbb{P}^1) \to \mathrm{Lie}(\mathcal{X}'/\mathbb{P}^1) \to 0$$

by Theorem 2.4 (also see `https://haochencheng1.github.io/restricted_lie_algebra.pdf`). In particular, this is

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-p) \to \mathrm{Lie}(\mathcal{X}'/\mathbb{P}^1) \to 0,$$

which is clear a contradiction.

5.2. MMV does not imply IMV in characteristic p.

Proposition 5.2. MMV does not imply IMV in characteristic p.

The example is given by Helm [Hel22], where they constructed a smooth proper unitary Shimura curve Y_U such that it is generically ordinary for some p. Moreover, the generic fibre X/K satisfies $\mathrm{Tr}_{K/k}(X) = 0$, where $K := K((Y_U)_{\mathbb{F}_p})$. And it satisfies the property that for some field extension K'/K, $X^t((K')^{\mathrm{perf}})$ is not finitely generated, where $(K')^{\mathrm{perf}} = (K')^{p^{-\infty}}$ is the maximal purely inseparable extension of K'.

According to [Rös15, Theorem 1.1] and [Rös20, Appendix B], if C is a smooth proper curve defined over a perfect field k of characteristic $p > 0$ and $\mathcal{X}/C$ is an abelian scheme (not necessarily principally polarized) with ample Hodge bundle. Then $X(K^{\mathrm{perf}}) = X(K^{p^{-m}})$ for some $m \gg 0$. In particular, if $\mathrm{Tr}_{K/k}(X) = 0$, then $X(K^{\mathrm{perf}})$ is finitely generated by Lang-Néron theorem (see [Con06]).

Let $C \coloneqq (Y_U)_k$ be the reduction to k as above and let $f : \mathcal{X} \to C$ be the universal family. So, for some finite covering $\pi : C' \to C$ with $K(C') = K'$, $X^t_{K'}((K')^{\mathrm{perf}})$ is not finitely generated. Since X and X^t are isogenous, $X_{K'}((K')^{\mathrm{perf}})$ is also not finitely generated. Let $\mathcal{E} \coloneqq f_*\Omega_{\mathcal{X}/C}$ be the Hodge bundle of $\mathcal{X}/C$, it follows that $\pi^*\mathcal{E}$ is not ample. So, $\mathcal{E}$ is not ample (see [Laz04, Proposition 6.1.2]).

Together with [Rös15, Theorem 1.2], $\bar{\mu}_{\min}(\mathcal{E}) = 0$ and one can argue as in [Rös15] to show that $\mathrm{Lie}(\mathcal{X}/C)$ has a subbundle V satisfying $(F^n_C)^*V \cong V$ for any $n \geq 1$. By Lange-Stuhler theorem, there exists a finite étale morphism $g : D \to C$ such that $g^*V \cong \mathcal{O}_D^{\oplus m}$ is a trivial bundle. This then gives a trivial subbundle $\mathcal{O}_D^{\oplus m} \subseteq \mathrm{Lie}(\mathcal{X}_D/D)$. To draw a conclusion, one can argue as in [Hel22, Lemma 4.1] that $X_{K(D)}$ also has trivial $K(D)/k$-trace, as the proof only depends on the geometric points. In particular, this shows that the trivial vector bundle g^*V does not come from a constant abelian scheme over D.

6. Some remarks and further question

In this section, we explain the reason that we can only study the maximal variation of abelian schemes using positivity of Hodge bundles at the level of global sections of Lie algebra.

6.1. **Positivity of Hodge bundles and maximal variation of abelian schemes.** We show that the non-ampleness of the Hodge bundle does not imply the existence of an isotrivial factor.

In characteristic p, an example is the same one in Proposition 5.2.

In characteristic 0, an example was introduced by Faltings in his famous paper [Fal83, Section 5]. He shows that there exists a family of abelian varieties $\mathcal{X}$ over a complex smooth proper curve C with relative dimension 8 such that it's non-rigid and has no isotrivial factors.

Actually, it was known to be the first case of non-rigid abelian schemes over complex smooth (projective) curve by Saito [Sai93]: all such abelian schemes of relative dimension less than 8 would be rigid.

Let $\mathcal{X}/C$ be the Faltings' example and write $f_*\Omega_{\mathcal{X}/C} \cong \mathcal{A} \oplus \mathcal{U}$ be the Fujita decomposition as before. In particular, the degree-zero part $\mathcal{U} \subset f_*\Omega_{\mathcal{X}/C}$ does not arise from an isotrivial abelian subscheme.

Moreover, it cannot arise from an isotrivial group subscheme either. Indeed, since $\mathcal{X}/C$ is projective, any closed group subscheme of $\mathcal{X}$ is also projective over C. An isotrivial closed group subscheme must therefore be a finite group scheme over C by Cartier's theorem, which says that any finite type group scheme over a characteristic 0 field is smooth.

Such a group scheme has trivial Lie algebra at every point, and hence cannot give rise to the non-trivial unitary bundle $\mathcal{U}$ through the integration process.

We believe that this gives a full answer to Starr and Rössler in `https://mathoverflow.net/questions/166193/ampleness-of-hodge-bundles-over-complex-curves`.

6.2. **A conjecture in characteristic p.** On the other hand, we raised the following conjecture in [Che26].

Conjecture 6.1 ([Che26])**.** Let $\mathcal{X}/C$ be a principally polarized abelian scheme. If $f_*\Omega_{\mathcal{X}/C}$ is non-nef, then $\mathrm{Tr}_{K/k}(X) \neq 0$.

The evidence for Conjecture 6.1 is that both the statements: non-nefness of Hodge bundle and non-zero trace (after assuming GMV, but one can then deduce the general case as in [Che26, Section 5]) are purely positive characteristic phenomenons. And the known cases with non-nef Hodge bundle all have trace non-zero (see [Che26]).

If Conjecture 6.1 fails, then the positivity of Hodge bundle will be a more mysterious object to study in positive characteristic, and we won't expect its relation to the study maximal variation of abelian varieties anymore. We do not know how to either do this or find a counterexample at this point.

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