We are writing this notes because we want to give a generalization of Chebotarev density theorem for a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$.

1. The classic Chebotarev density theorem

1.1. **Preliminaries of number theorey.** A number field K is a finite extension of $\mathbb{Q}$. We denote its ring of integers $\mathcal{O}_L$, which is the integral closure of $\mathbb{Z}$ in K, and is a Dedekind domain.

Let L/K be a fintie extension of degree n of number fields. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Then the basic commutative algebra tells us that

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_i}...\mathfrak{P}_g^{e_g}$$

for some prime ideals $\mathfrak{P}_j$ of $\mathcal{O}_L$.We call e_j the ramification number of $\mathfrak{P}_j|\mathfrak{p}$.

Let $f_j := [\mathcal{O}_L/\mathfrak{P}_j : \mathcal{O}_K/\mathfrak{p}]$, then we have a equality

$$n = \sum_j e_j f_j$$

by degree chasing.

Moreover, if we let L/K be a Galois extension, then $G := \mathrm{Gal}(L/K)$ acts transitively on the set $\{\mathfrak{P}_1, ..., \mathfrak{P}_g\}$. This implies that if L/K is Galois, the number e_j and f_j are independently of j, and only depend on $\mathfrak{p}$. So, we have

$$n = efg,$$

and

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e}...\mathfrak{P}_g^{e}$$

.

Let $\mathfrak{P}$ be such a prime with $\mathfrak{P}|\mathfrak{p}$. We can define

$$D_{\mathfrak{P}}(L/K) := \{\sigma \in G; \sigma(\mathfrak{P}) = \mathfrak{P}.\},$$

called the decomposition group of $\mathfrak{P}$ in L/K.

In other words, it is the stabilizer group of G acts on the group set $\{\mathfrak{P}_1, ..., \mathfrak{P}_g\}$. As this action is transitive, $D_{\mathfrak{P}}$ has index g, and hence of order ef for any $\mathfrak{P}|\mathfrak{p}$.

Theorem 1.1. Let $k := \mathcal{O}_K/\mathfrak{p}$ and $l := \mathcal{O}_L/\mathfrak{P}$ If L/K is Galois, the so is l/k, and we have a surjection

$$D_{\mathfrak{P}}(L/K) \twoheadrightarrow \mathrm{Gal}(l/k)$$

with kernel $I_{\mathfrak{P}}(L/K)$, called the inertial group.

So, $I_{\mathfrak{P}}$ has order e for any $\mathfrak{P}|\mathfrak{p}$.

Moreover, given an umramified prime $\mathfrak{p}$ of $\mathcal{O}_K$, i.e., a prime $\mathfrak{p}$ with decomposition

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1...\mathfrak{P}_g$$

with $e = 1$.

We then have an isomoprhism of groups

$$D_{\mathfrak{P}}(L/K) \cong \mathrm{Gal}(l/k)$$

for each $\mathfrak{P}|\mathfrak{p}$. Let $\mathrm{Frob}_{\mathfrak{P}}$ denote the Frobenius in $\mathrm{Gal}(l/k)$ and consider it as an element in $D_{\mathfrak{P}}(L/K)$, hence in $\mathrm{Gal}(L/K)$. Since all $D_{\mathfrak{P}}$ are conjugated to each other, this element $\mathrm{Frob}_{\mathfrak{P}}$ is well-defined up to conjugacy. And we shall call it the Froebnius substitution, denoted by $\mathrm{Frob}_{\mathfrak{p}}$ or $(\mathfrak{p}, L/K)$.

1.2. **Chebotarev density theorem.** The Chebotarev density theorem in algebraic number theory describes statistically the splitting of primes in a given Galois extension K of the field $\mathbb{Q}$ of rational numbers.

Theorem 1.2 (Chebotarev density theorem)**.** Let L be a finite Galois extension of a number field K with Galois group G. Let $\mathcal{C}$ be a conjugacy class in G. Let

$$T := \{\mathfrak{p}; \mathfrak{p} \text{ unramified in } L, (\mathfrak{p}, L/K) = \mathcal{C}\}.$$

Then T has Dirichlet density

$$\delta(T) = \frac{\#\mathcal{C}}{\#G}.$$

We want to explain the classical ways to define all these concepts.

Definition 1.3 (Dirichlet density)**.** Let T be a set of primes of K. If there exists a $\delta(T)$ such that

$$\delta(T) = \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in T}(\#\mathcal{O}_K/\mathfrak{p})^{-s}}{-\log(s-1)},$$

then we call $\delta(T)$ the Dirichlet density of T.

We now want to show the denominator

$$-\log(s-1) \sim \sum_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)} (\#\mathcal{O}_K/\mathfrak{p})^{-s}, s \to 1^+,$$

so that we have the right definition of "density".

At here, let $f(s), g(s)$ be two functions defined for $s > 1$ and real, then we write

$$f(s) \sim g(s), s \to 1^+,$$

if $f(s) - g(s)$ is bounded for

$$1 < s < 1 + \epsilon, s \in \mathbb{R},$$

for some $\epsilon > 0$.

We consider the prime zeta function of the number field K,

$$\zeta_K(s) = \Pi_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)} \frac{1}{1-(\#\mathcal{O}_K/\mathfrak{p})^{-s}}.$$

It's a fact that $\zeta_K(s)$ is meromorphic on a neighborhood of 1 and having a pole of order 1 at 1. So, we can write

$$\zeta_K(s) = \frac{a}{s-1} + g(s), \qquad (*)$$

for some a and holomorphic function $g(s)$ in a neighborhood of 1. Moreover, $a > 0$ since $\zeta_K(s) > 0$ for $s > 1$ and real.

Lemma 1.4. Let $u_1, u_2, ...$ be a sequence of real numbers ≥ 2 such that

$$f(s) := \Pi_{j=1}^{\infty} \frac{1}{1-u_j^{-s}}$$

is uniformally conversent on each region

$$D(1, \delta, \epsilon) = \{\Re(s) \geq 1 + \delta; |\arg(s-1) \leq \frac{\pi}{2} - \epsilon|\}, \delta, \epsilon > 0.$$

Then

$$\log(f(s)) \sim \sum \frac{1}{u_j^s}, s \to 1^+.$$

Proof. We have

$$\log f(s) = \sum_{j=1}^{\infty} \log \frac{1}{1-u_j^{-s}} = \sum_j \sum_{m=1}^{\infty} \frac{1}{m u_j^{sm}} = \sum_j \frac{1}{u_j^s} + \sum_j \sum_{m=2}^{\infty} \frac{1}{m u_j^{sm}} = \sum_j \frac{1}{u_j^s} + g(s),$$

where
$$|g(s)| \leq \sum_{j=1}^{\infty}\sum_{m=2}^{\infty}|\frac{1}{mu_j^{sm}}| = \sum_{j=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{mu_j^{\Re(s)m}}.$$

Since $u \geq 2, \Re(s) > 1$,
$$\sum_{m=2}^{\infty}\frac{1}{mu^{m\Re(s)}} \leq \sum_{m=2}^{\infty}\frac{1}{2}(\frac{1}{u^{\Re(s)}})^m = \frac{u^{-2\Re(s)}}{2(1-u^{-\Re(s)})} < \frac{u^{-2\Re(s)}}{1-u^{-2\Re(s)}}.$$

Hence
$$|g(s)| \leq \sum_{j=1}^{\infty}\frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}.$$

Since $f(s)$ is holomorpic in a neighborhood with $\Re(s) > 1$, $f(2s)$ is holomorphic for $\Re(s) > 1/2$, hence bounded near $s = 1$. Since $\sum_{j=1}^{\infty}\frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}$ converges absolutely if and only if $\Pi_j(1+\frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}) = f(2\Re(s))$ converges absolutely, we know that $g(s)$ is bounded as $s \to 1^+$.

□

As a result, we can take logs to (*), and we find that
$$\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)}(\#\mathcal{O}_K/\mathfrak{p})^{-s} \sim \log(\zeta_K(s)) \sim \log(\frac{1}{s-1}), s \to 1^+,$$
which is what we need.

We want to give an application of the Chebotarev density theorem. We will talk about two extreme spliting types.

Let $\mathrm{Spl}(L/K)$ to denote the primes $\mathfrak{p}$ in $\mathcal{O}_K$ that splits completely in $\mathcal{O}_L$.

Lemma 1.5. $\delta(\mathrm{Spl}(L/K)) = \frac{1}{[L:K]}$

Proof. Let $\mathfrak{p}$ be an unramified prime of $\mathcal{O}_K$. Then $\mathrm{Frob}_{\mathfrak{p}} = 1$ if and only if $\mathfrak{p}$ splits completely. By 1.2, we know
$$\delta(\mathrm{Spl}(L/K)) = \frac{1}{[L:K]}.$$

□

Let $\mathrm{Inert}(L/K)$ be the set of primes $\mathfrak{p}$ in $\mathcal{O}_K$ that is inert, i.e., $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}$ for one prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$.

Then we see that $\mathfrak{p}$ is inert if and only if $\mathrm{Frob}_{\mathfrak{p}}$ generates the whole group G. In other words, if L/K is a cyclic extension, then by 1.2,
$$\delta(\mathrm{Inert}(L/K)) = \frac{\varphi([L:K])}{[L:K]}.$$

And if L/K is not cyclic,
$$\delta(\mathrm{Inert}(L/K)) = 0.$$

2. Review of étale local system and étale cohomology

2.1. **étale local system and lisse sheaves.** In this section we want to review the Weil Conjecture and Grothendieck-Lefschetz trace formula for vaireties over finite fields, which plays the role as the L-function. We assume the basic knowledge about étale site and étale fundamental group. In particular, given a scheme X over k with a geometric point $x \in X$, we have the short exact sequence
$$0 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, x) \longrightarrow \pi_1^{\text{ét}}(X, x) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 0.$$

Definition 2.1 (Locally constant étale sheaf)**.** Let X be a scheme, and let $\mathcal{F}$ be a sheaf of sets on $X_{\text{ét}}$. We say $\mathcal{F}$ is locally constant if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}_{|U_i}$ is a constant sheaf.

For example, for any prime $l \neq \text{char}(k)$, one can consider the constant sheaf $\underline{\mathbb{Z}/l^n\mathbb{Z}}$ for any $n \geq 1$. And the question is that give an étale covering $\{U_i \to X\}$, how does this glue to a locally constant sheaf, which is fibrewise $\mathbb{Z}/l^n\mathbb{Z}$. This is well-explained by using the étale fundmental group and monodromy theory, which is an analogue to the covering space theory in the manifold case.

Definition 2.2 (Locally constant constructibleétale sheaf)**.** A locally constant constructible étale sheaf is a locally constant étale sheaf, whose fibrewise is a finite set.

From now, we will use $\text{LCC}(X)$ to denote the category of locally constant constructible étale sheaf on X.

Theorem 2.3. *Let $X' \to X$ be a finite étale cover of X. Then the functor of points sheaf*

$$\underline{X'} := \text{Hom}_X(-, X')$$

on $X_{\text{ét}}$ is a LCC sheaf. Moreover, the association $X' \mapsto \underline{X'}$ gives an equivalence of cateogries

$$\{\text{finite étale covers of } X\} \leftrightarrow \text{LCC}(X)$$

Proof. The opposite direction is to use the Étale space of a sheaf, which is a finite étale covering of X. ☐

On the other hand, as in the real manifold cases, the finite étale cover of X corresponds to how the fundmental group acting on the fibres. This should be obvious, depending on how the reader realize the étale fundamental group.

Indeed, given a geometric point $x \in X$, one can define the fibre functor

$$\text{Fib}_x = \varinjlim \text{Hom}_X((U, u), -).$$

The action of $\pi_1^{\text{ét}}(X, x)$ on the finite set $\text{Fib}_x(X')$ is continuous with $\text{Fib}_x(X')$ given the discrete topology, and hence Fib_x gives a natural functor

$$\{\text{finite étale covers of } X\} \longrightarrow \{\text{finite discrete } \pi_1^{\text{ét}}(X, x)\text{-sets}\}.$$

Theorem 2.4 (Grothendieck)**.** *Fib_x is an equivalence of categories.*

Corollary 2.5. *Let $X = \text{Spec}(k)$, then we have a equivalence of categoeis*

$$\{\text{étale } k\text{-schemes}\} \leftrightarrow \{\text{finite (discrete) } \text{Gal}(\bar{k}/k)\text{-sets}\}.$$

Also notice that it preserves group structures, so we also have a equivalence between étale k-group schemes ad the category of finite (discrete) $\text{Gal}(\bar{k}/k)$-groups.

Afterall, by 2.3 and 2.4, we can see the locally constant constructible étale sheaf corresponds to the finite $\pi_1^{\text{ét}}(X, x)$-sets S, which corresponds to a continuous representation

$$\pi_1^{\text{ét}}(X, x) \longrightarrow \text{Aut}(S).$$

Also, as $\Gamma(X, \mathcal{F}) = \text{Hom}_{\text{Sh}}(\underline{\mathbb{Z}}, \mathcal{F})$. If $\mathcal{F}$ is locally constant constructible with fibrewise S, then the latter group equals to $\text{Hom}_{\pi_1^{\text{ét}}}(\mathbb{Z}, S)$ by 2.3 and 2.4. It then equals to $S^{\pi_1^{\text{ét}}}$, the invariants of $\pi_1^{\text{ét}}$. Notice that this is exactly the same proof as in the manifold cases.

So far, we only talked about the fintie set S, such as $\mathbb{Z}/l^n\mathbb{Z}$. We want to push this idea further. And we need to introduce the idea of l-adic sheaves.

Definition 2.6 (l-adic sheaf)**.** An l-adic sheaf M is a family $(M_n, f_{n+1} : M_{n+1} \to M_n)$ such that 1. for each n, M_n is a constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules;

2. for each n, the map $f_{n+1} : M_{n+1} \to M_n$ induces an isomoprhism $M_{n+1}/l^n M_{n+1} \to M_n$.

The l-adic sheaf M is said to be locally constant (also called a lisse sheaf)if each M_n is locally constant. So, it corresponds to a continuous representation

$$\rho : \pi_1^{\text{ét}}(X, x) \to \mathrm{Aut}(M).$$

And we can define the cohomology groups

$$H^r_{et}(X, M) = \varprojlim_n H^r_{et}(X, M_n).$$

In particular,

$$H^r_{et}(X, \mathbb{Z}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}).$$

Here is one reason that we have to define the l-adic sheaf in this way is that intuitively, we want to define $H^1(X, \mathbb{Z}_l) = \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}_l)$. But since $\pi_1^{\text{ét}}(X, x)$ has profintie topology, while $\mathbb{Z}_l$ has discrete topology, any continuous homomorphism f must map an open subgroup of $\pi_1^{\text{ét}}$ to 0. Hence f will factor through a finite group, which forces itself to zero.

However, when one consider $\mathbb{Z}_l$ as a lisse sheaf,

$$H^1_{et}(X, \mathbb{Z}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}) = \varprojlim_n \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}/l^n\mathbb{Z}) \cong \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}_l),$$

where the last $\mathbb{Z}_l$ carries the l-adic topology.

Similarly, we define a sheaf of $\mathbb{Q}_l$-vector spaces as a lisse sheaf $M = (M_n)$, with

$$H^r_{et}(X, M) = (\varprojlim_n H^r_{et}(X, M_n)) \otimes \mathbb{Q}_l.$$

In particular,

$$H^r_{et}(X, \mathbb{Q}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}) \otimes \mathbb{Q}_l.$$

As a summary, we have that given a $\mathbb{Q}_l$-sheaf $\mathcal{F}$ is the same giving a continuous represetation

$$\rho : \pi_1^{\text{ét}}(X, x) \to \mathrm{GL}(V),$$

where $V = \mathcal{F}_x \cong \mathbb{Q}_l^n$ is the fibre of $\mathcal{F}$ at x.

At last, we give the definition of Tate twist.

Definition 2.7 (Tate twist)**.** Let $\mathbb{Z}_l$ denote the trivial $\mathbb{Z}_l$-sheaf. Then $\mathbb{Z}_l(1)$ is defined as the $\mathbb{Z}_l$-sheaf associated to the lisse sheaf $T_l\mu_{l^\infty} = \varprojlim \mu_{l^n}(k^s) \cong \mathbb{Z}_l^*$ over $\mathrm{Spec}(k)$.

In particular, if $k = \mathbb{F}_q$ is a finite field, then $\mathbb{Q}_l(1)$ associates to the l-adic cyclotomic character

$$\rho : \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to \mathbb{Z}_l^*, \mathrm{Frob}_q \mapsto q.$$

We also define $\mathbb{Z}_l(-1) = \mathbb{Q}_l(1)^\vee$. For any l-adic lisse sheaf $\mathcal{F}$, we define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(1)^{\otimes d}$.

2.2. **Grothendieck-Lefschetz trace formula.** In geometry, Lefschetz fixed point theorem is a classical tool to help us check the fixed points of a good map by using the linear data.

Theorem 2.8 (Lefschetz fixed point theorem)**.** Let M be a compact, oriented manifold. Let $f : M \to M$ be a continuous map with isolated fixed points. Then

$$\#\{x \in M : f(x) = x\} = \sum_i (-1)^i \mathrm{trace}(f^* | H^i(M, \mathbb{R})).$$

Inspired by this, Grothendieck has an analogue applied to a variety X over a finite field $k = \mathbb{F}_q$, with $q = p^m$ and $\pi_X := F^m_{X/\mathbb{F}_q} : X \to X$ is the geometric Frobenius.

Definition 2.9 (cohomology with compact support)**.** For any torsion sheaf $\mathcal{F}$ on a variety U, we define

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F}),$$

where X is any complete variety containing U as a dense open subvariety and j is the inclusion map.

Theorem 2.10 (Grothendieck-Lefschetz trace formula)**.** Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf over X. Then

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{trace}(\pi_x|\mathcal{F}_x) = \sum_i (-1)^i \mathrm{trace}(\pi_X^*|H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F})),$$

where $\mathcal{F}_x$ is the fibre at x, and π_x is the geometric frobenius at k_x, i.e., the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_q/k_x)$.

In particular, if $\mathcal{F} = \mathbb{Q}_l$ is the constant $\mathbb{Q}_l$-sheaf, the left hand side becomes $\sum_{x \in X(\mathbb{F}_q)} 1 = \#X(\mathbb{F}_q)$. Since $X(\mathbb{F}_q) = \{x \in X(\bar{\mathbb{F}}_q); \pi_X(x) = x\}$, this means we have an analogue and generalization of the Lefschetz fixed point theorem.

Corollary 2.11. Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf over X. Then

$$\#X(\mathbb{F}_{q^m}) = \sum_i (-1)^i \mathrm{trace}(\pi_X^m|H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l)).$$

2.3. **Weil Conjectures.** The trace formula 2.10 inspired Weil that we can use the linear data given by the l-adic cohomology to study the original geometric object. Now, let X be a smooth projective variety of dimension d over $\mathbb{F}_q$.

We can define the zeta function of X by

$$Z(X,t) := \exp(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}) \in \mathbb{Q}[[t]].$$

Alternatively, let $|X|$ be the set of closed points of X. Let $x \in |X|$ be a closed point with residue field k_x. Then

$$Z(X,t) = \Pi_{x \in |X|}(1 - t^{[k_x:\mathbb{F}_q]})^{-1}.$$

One can also define the arithmetic zeta function

$$\zeta_X(t) = \Pi_{x \in |X|}(1 - \#k_x^{-t})^{-1}.$$

Then one can check that $Z(X, q^{-t}) = \zeta_X(t)$.

Note that the arithmetic zeta function is just a generalization of the classic zeta functions for scheme of fintie type over $\mathbb{Z}$. For example, if K is a number field and $\mathcal{O}_K$ its ring of integers, then the Dedekind zeta function is defined as

$$\zeta_K(s) = \Pi_{\mathfrak{p} \subset \mathcal{O}_K}(1 - \#(\mathcal{O}_K/\mathfrak{p}))^{-1}.$$

We want to use the general fact that for an endomorphism φ of a finite dimensional vector space V over a field K, we have an identity of a formal power series

$$\exp(\sum_{n=1}^{\infty} \mathrm{trace}(\varphi^n; V) \cdot \frac{t^n}{n}) = \det(Id - t \cdot \varphi; V)^{-1}. \tag{1}$$

Using 2.11 and (1),

$$Z(X,t) = \exp(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}) = \exp(\sum_{n=1}\sum_i (-1)^i \mathrm{trace}(\pi_X^n|H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l)) \cdot \frac{t^n}{n})$$

$$= \Pi_{i=0}^{2d} \exp(\sum_{n=1} \mathrm{trace}(\pi_X^n; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)) \cdot \frac{t^n}{n})^{(-1)^i} = \Pi_{i=0}^{2d} \det(Id - t \cdot \pi_X; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l))^{(-1)^{i+1}}.$$

Let $P_i(t) := \det(id - t \cdot \pi_X; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l))$. Suppose it has decomposition over $\mathbb{C}$: $P_i(t) = \Pi_j(1 - \alpha_{ij}t)$. Then α_{ij} are eigenvalues of π_X on $H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$.

Theorem 2.12 (Known as Weil Conjecture, now a theorem of Deligne)**.** Let X be a non-singular projective variety of dimension d over $\mathbb{F}_q$, then

1. (Rationality) The zeta function of V is rational, of the form

$$Z(X,t) = \frac{P_1(t) \cdots P_{2d-1}(t)}{P_0(t) \cdots P_{2d}(t)},$$

where $P_0(t) = 1 - t, P_{2d} = 1 - q^d t$, and each P_i is a polynomial with integer coeffients factoring over $\mathbb{C}$, as $P_i(t) = \Pi_j(1 - \alpha_{ij} t)$.

2. (Functional equation) The zeta function satisfies

$$Z(X, \frac{1}{q^d t}) = \pm q^{\frac{d \cdot \chi(X)}{2}} \cdot t^{\chi(X)} \cdot Z(X, t),$$

where $\chi(X)$ is the Euler chracteristic of X.

3. (Riemann hypothesis) The roorts α_{ij} are algebraic integers of complex absolute value $|\alpha_{ij}| = q^{i/2}$.

By our discussion before the theorem, it means that the eigenvalues of π_X on $H^i_c(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$ are algebraic integers of complex absolute values $|\alpha_{ij}| = q^{i/2}$.

We give two basic example:

Example 1: Consider the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. Then $H^0(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = H^2(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = \mathbb{Q}_l$ and $H^1(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = 0$. We may take the convention that the determinant of endomorphism on trivial vector space is one. Then

$$Z(\mathbb{P}^1, t) = \frac{1}{(1-t)(1-qt)}.$$

On the other hand, for any n,

$$\#\mathbb{P}^1(\mathbb{F}_{a^n}) = q^n + 1.$$

So, $Z(\mathbb{P}^1, t) = \exp(\sum_{n \geq 1} \frac{q^n+1}{n} t^n)$.

Since $\sum_{n \geq 1} \frac{q^n+1}{n} t^n = -\log(1 - qt) - \log(1 - t)$, we have the same result.

Example 2: Let X be an abelian varietiey of dimension g over $\mathbb{F}_q$. Then $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) \cong \wedge^i H^1(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$. This shows that the eigenvalues α_{ij}'s have the relation that α_{ij} are exactly of the form $\alpha_{ij} = \Pi_{j' \in I, I \subset \{1, \ldots, 2g\}, \#I = i} \alpha_{1j'}$. So, the well study of Tate modules of abelian varieties give all information of the zeta functions.

Moreover, Deline later generalized his result to a general lisse sheaf $\mathcal{F}$ of any weight.

Definition 2.13 (weights)**.** Let $n \in \mathbb{Z}$. Let $F_x := \mathrm{Frob}_x^{-1}$.

1. A locally constant sheaf $\mathcal{F}$ of $\mathbb{Q}_l$-vector spaces on X is said to have weight n if for all closed points x of X, each eigenvalue of $F_x : \mathcal{F}_x \to \mathcal{F}_x$ is an algebraic number whose complex conjugates have absolute value $q^{[k_x:\mathbb{F}_q]n/2}$.

2. A fintie-dimensional $\mathbb{Q}_l$-vector space E endowed with a continuous action of $\pi_1^{\text{ét}}(X)$ has weight n, if , for all closed point x of X. each eigenvalue of $\mathrm{Frob}_x : E_x \to E_x$ is an algebraic number whose complex conjugates have absolute value $q^{-[k_x:\mathbb{F}_q]n/2}$.

Example: $\mathbb{Q}_l(1)$ is acted by Frob_q by sending it to q. So, it has weight -2.

Theorem 2.14 (Deligne)**.** Let $f : X \to S$ be a morphism of schemes of finite type over $\mathbb{F}_q$, and let $\mathcal{F}$ be a mixed sheaf of weight $\leq n$ on $X_{\bar{\mathbb{F}}_q}$. Then, for each i, the sheaf $R^i f_! \mathcal{F}$ on S is mixed of weight $\leq n + i$.

In particular, let $S = \mathrm{Spec}(\mathbb{F}_q)$, then for any lisse sheaf $\mathcal{F}$ of weight 0, the eigenvalues of π_X on $H^i_c(X_{\bar{\mathbb{F}}_q}, \mathcal{F})$ are algebraic numbers of complex absolute values $\leq q^{i/2}$.

And he showed it's equality when X is a curve.

Theorem 2.15 (Deigne)**.** Let X be a smooth proper curve over $\mathbb{F}_q$, $j : U \to X$ the inclusion of a dense open subset, and $\mathcal{F}$ a lisse sheaf that is pointwise pure of weight n on U. Then $H^i(X_{\bar{\mathbb{F}}_q}, j^* \mathcal{F})$ is pure of weight $n + i$.

2.4. **Poincaré duality.** We now want to finish this section by introducing another important and useful tool: the Poincaré duality. Let k be an algebraic closed field and let $\lambda = \mathbb{Z}/n\mathbb{Z}$ for some n prime to the characteristic of k.

We let $\Lambda(1)$ denote μ_n and $\Lambda(m) = \mu_n^{\otimes m}$. Let $\mathcal{F}^\vee(m) = \mathrm{Hom}(\mathcal{F}, \Lambda(m))$ be the hom sheaf.

Theorem 2.16 (Poincaré duality)**.** Let X be a smooth variety of dimension d over an algebraically closed field k.

1. There is a unique map $\eta(X) : H_c^{2d}(X, \Lambda(d)) \to \Lambda$ by sending $cl(P)$ to 1 for any closed point P on X, where $cl(P)$ is the image of 1 under the composition of the Gysin map (which is an isomorphism)

$$H^0(P, \Lambda) \to H_P^{2d}(X, \Lambda(d))$$

and the natural map

$$H_P^{2d}(X, \Lambda(d) \to H_c^{2d}(X, \Lambda(d))).$$

$\eta(X)$ is an isomorphism and called the trace map.

2. For any locally constant sheaf $\mathcal{F}$ of Λ-modules, there are canonical pairings

$$H_c^r(X, \mathcal{F}) \times H^{2d-r}(X, \mathcal{F}^\vee(d)) \to H_c^{2d}(X, \Lambda(d)) \cong \Lambda,$$

which are perfect pairings of finite groups.

As a result, if we let $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$ and take the limits, we have

$$H_c^r(X, \mathcal{F}) \cong H^{2d-r}(X, \mathcal{F}^\vee(d))^\vee$$

for every l-adic lisse sheaf $\mathcal{F}$.

3. Dirichlet density in dimensions ≥ 1

3.1. **Dirichlet density.** From now we fix a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$. This covers the example of number rings or a variety over a finite field. Let $d := \dim(X)$ and $|X|$ the set of all closed points of X.

If we suppose k is the constant field of X, then $|X|$ equals to the set of all points $x \in X$, with $[k_x : k] \leq \infty$. Moreover, there is a natural bijection

$$|X| \xrightarrow{\cong} X(\bar{k})/\mathrm{Gal}(\bar{k}/k),$$

where the target is the Galois orbits of $\bar{k}$-geometric points in X. In particular, given a closed point $x \in X$ with $[k_x : k] = n$ and k_x/k Galois, it corresponds to a $\mathrm{Gal}(k_x/k)$-orbit $X(k_x)/\mathrm{Gal}(k_x/k)$, which corresponds to exactly n primitive points in $X(k_x)$. At here, a primitive point $x \in X(k_x)$ means a point whose residue field is exactly k_x.

From now, we may suppose $k = \mathbb{F}_q$ for some p-power q. But notice that all theorems of this section apply to characteristic zero cases, with some minor changes in the proofs need to be fixed. We use q_x to denote $\#k_x$ for any closed point $x \in X$. Let Frob_x denote the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_q/k_x)$.

Theorem 3.1. Let $f : X \to Y$ be a proper and smooth morphism of schemes of finite type over $\mathrm{Spec}\mathbb{Z}$.

1. Suppose that all fibres of f have dimension $\leq m$, then there exists a constant $C > 0$ such that for all $y \in |Y|$ and all $n \geq 1$, we have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \leq \frac{1}{n} \cdot C \cdot q_y^{nm};$$

2. Suppose that f is surjective and all fibres are geometrically irreducible of dimension $m > 0$. Then there exists a constant $C' > 0$ such that for all $y \in |Y|$ and all $n \geq 1$, we have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \geq \frac{1}{n} \cdot (q_y^{nm} - C' q_y^{n(m-\frac{1}{2})}).$$

Proof. Let X_y denote the fibre of f above y and $k_y^{(n)}$ an extension of k_y of degree n. Then by 2.11, we know

$$\#X_y(k_y^{(n)}) = \sum_{i=0}^{2m} (-1)^i \mathrm{trace}(\mathrm{Frob}_y^n | H_c^i(X_y \times_{k_y} \bar{k}_y), \mathbb{Q}_l).$$

And by 2.12, we know the eigenvalues of Frob_y on $H_c^i(X_y \times_{k_y} \bar{k}_y, \mathbb{Q}_l)$ are algebraic numbers of complex absolute value $\leq q_y^{i/2}$. Moreover, by constructibility and (smooth) proper base change, $R^i f_* \mathbb{Q}_l$ is locally constant and the total number of eigenvalues is bounded independently of y.

So,

$$\#X_y(k_y^{(n)}) = |\#X_y(k_y^{(n)})| \leq C \cdot q_y^{nm} \tag{2}$$

for some constant C.

As we discussed before, any $x \in |X|$ with $f(x) = y$ and $[k_x : \mathbb{F}_q] = n$ corresponds to exactly n primitive points in $X_y(k_y^{(n)})$. This applies (1).

On the other hand, to show the lower bound, we have to give a bound for the non-primitive points in $X_y(k_y^{(n)})$. At here, the non-primitibe points are those in $X_y(k_y^{(n)})$, whose definition field is stirctly contained in $k_y^{(n)}$.

By (2), we have

$$\#(X_y(k_y^{(n)})^{\mathrm{nonprim}}) \leq \sum_{n'|n, m \neq n} \#(X_y(k_y^{(n')})) \leq \sum_{1 \leq n' \leq \frac{n}{2}} C \cdot q_y^{n'm} = C \cdot \frac{q_y^{\lfloor n/2 \rfloor m} - 1}{1 - q_y^{-m}} \leq 2C q_y^{nm/2}. \tag{3}$$

Still by 2.12, we know $H_c^{2m}(X_y \times_{k_y} \bar{k_y}, \mathbb{Q}_l)$ has dimension 1 and the eigenvalue of Frob_y is equal to q_y^m.

We then have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} = \frac{1}{n} \cdot \#X_y(k_y^{(n)})^{\mathrm{prim}}$$

$$\geq \frac{1}{n} \cdot (\#(X_y(k_y^{(n)})) - 2Cq_y^{nm/2}) \geq \frac{1}{n} \cdot (q_y^{nm} - Cq_y^{n(m-1/2)} - 2Cq_y^{nm/2})$$

$$\geq \frac{1}{n} \cdot (q_y^{nm} - 3Cq_y^{n(m-1/2)}).$$

□

We now want to define the Dirichlet density of points, which is a generalization of number field cases. 3.1 give the well-behaved bounds for us to show the Dirichlet density is well-behaved.

Let X be a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and of dimension $d > 0$. Then for any subset $S \subset |X|$ and a complex parameter s, we can define

$$F_S(s) = \sum_{x \in S} q_x^{-s}. \tag{4}$$

For example, when $X = \mathrm{Spec}\mathbb{Z}$ and $S = |X|$, it's just the prime zeta function

$$F_{|X|}(s) = \sum_p \frac{1}{p^s} \sim \log(\frac{1}{s-1}), s \to 1^+.$$

Lemma 3.2. 1. The series (4) converges absolutely and locally uniformly for $\Re(s) > d$. Thus it deines a holomorphic function in this region.

2. We have

$$\lim_{s \to d^+} F_{|X|}(s) = \infty,$$

where the limit is taken with s approaching d along the real line from the positive direction.

Proof. Let $K = K(X)$ be the function field of X and assume that $\mathrm{char}(K) > 0$. Let $\mathbb{F}_q$ be its field of constants. Note that it's sufficient to replace X by a Zariski dense open subscheme, where one wants to use Noetherian induction for (1).

We may suppose that X is a geometrically irreducible scheme over $Y = \mathrm{Spec}\mathbb{F}_q$. Then we have

$$F_S(s) = \sum_{n\geq 1} q^{-ns}\#\{x \in S; [k_x : \mathbb{F}_q] = n\}.$$

Taking absolute values and using 3.1 (1), we see that this series is dominated by

$$\sum_{n\geq 1} q^{-n\Re(s)} \cdot \frac{1}{n} C q^{nd} = C|\log(1 - q^{d-\Re(s)})|.$$

Clearly this is locally uniformly bounded in the area of $\Re(s) > d$.

For the second conclusion, we take $s \in \mathbb{R}$. By 3.1 (2),

$$F_{|X|}(s) = \sum_{n\geq 1} q^{-ns}\#\{x \in |X|; [k_x : \mathbb{F}_q] = n\} \geq \sum_{n\geq 1} q^{-ns}\frac{1}{n}(q^{nd} - C' q^{n(d-\frac{1}{2})})$$

$$= -\log(1 - q^{d-s}) + C'\log(1 - q^{d-s-\frac{1}{2}}).$$

This goes to infinity clearly.

□

The above Lemma also applies to characteristic zero cases, with Y changed by spectrum of number rings and the sum need to be done over all places of Y.

Definition 3.3 (Dirichlet density)**.** If the limit

$$\mu_X(S) = \lim_{s\to d^+} \frac{F_S(s)}{F_{|X|}(s)}$$

exists, we say that S has a Dirichlet density and $\mu_X(S)$ is called the Dirichlet density of S (in X).

Lemma 3.4. 1. If S has Dirichlet density, then $0 \leq \mu_X(S) \leq 1$.

2. The set $|X|$ has density 1.

3. If S is contained in a Zariski closed proper subset of X, then S has density 0.

4. If $S_1 \subset S \subset S_2 \subset |X|$ such that $\mu_X(S_1)$ and $\mu_X(S_2)$ exist and equals, then $\mu_X(S)$ exists and is equal to $\mu_X(S_1) = \mu_X(S_2)$.

5. For any subsets $S_1, S_2 \subset |X|$, if three of the following densities exist, then so does the fourth and we have

$$\mu_X(S_1 \cup S_2) + \mu_X(S_1 \cap S_2) = \mu_X(S_1) + \mu_X(S_2).$$

Proof. They are easy to see. For (3), S has smaller dimension, so $F_S(s)$ is bounded near $s = d$ by 3.2, while the denominator goes to infinity. □

Lemma 3.5. Let $f : X \to Y$ be a dominant morphism of integral schemes of fintie type over $\mathrm{Spec}(\mathbb{Z})$. Suppose that $\dim(X) = \dim(Y)$ and that f is totally inseparable at the generic point. Then any given subset $S \subset |X|$ has a density if and only if $f(S)$ has a density, and then $\mu_X(S) = \mu_Y(f(S))$.

Proof. We may choose a Zariski dense open subset $V \subset Y$ such that $U := f^{-1}(U) \to V$ is fintie and totally inseparable at every point. And we may replace X by U and Y by V. Then f induces isomorphisms on the residue fields, since they are all perfect. Hence we have $F_S(s) = F_{f(S)}(s)$ for any subset $S \subset |X|$. □

Theorem 3.6. Let $f : X \to Y$ be a moprhism of schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Suppose that X is integral and that f is non-constant. Then the set

$$\{x \in |X|; [k_x : k_{f(x)}] = 1\}$$

has Dirichlet density 1.

Proof. First we replace Y by the closure of the image of f. Next we write $d = \dim X, e = \dim Y, m = d - e$. Choose a Zariski dense open subset $U \subset X$ such that the fibre dimension $f_{|U} : U \to Y$ is everywhere equal to m. By the lemmas before, we may replace X by U. Now put

$$\{x \in |X|; [k_x : k_{f(x)}] \geq 2\}.$$

By 3.2 (2), we only need to show that $F_S(s)$ converges absolutely and uniformly near $s = d$. Taking absolute values and using 3.1, we see that the series is domonated by

$$\sum_{x \in S} q_x^{-\Re(s)} = \sum_{y \in |Y|} \sum_{x \in S, f(x) = y} q_y^{-[k_x : k_y] \cdot \Re(s)}$$

$$= \sum_{y \in |Y|} \sum_{n \geq 2} q_y^{-n\Re(s)} \cdot \#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \leq \sum_{y \in |Y|} \sum_{n \geq 2} q_y^{-n\Re(s)} \frac{1}{n} C q_y^{nm}$$

$$\leq \sum_{y \in |Y|} C \frac{q_y^{2(m - \Re(s))}}{1 - q_y^{m - \Re(s)}} \leq \frac{C}{1 - 2^{m - \Re(s)}} \cdot F_{|Y|}(2(\Re(s) - m)).$$

By 3.2 again, this converges locally uniformly for $Re(s) > m + \frac{e}{2} = d - \frac{e}{2}$. Since f is nonconstant, $e > 0$ and this converges uniformly near $s = d$. $\square$

The above theorem is a generalization to the fact that the set of primes of absolute degreee 1 in a number field K has Dirichlet density 1 (in K).

3.2. **Chebotarev density theorem.** We now can start to state and prove the Chebotarev density theorem. Let X be a scheme of fintie type over $\mathrm{Spec}(\mathbb{Z})$ as before. Let $\tilde{X} \to X$ be a finite étale Galois covering with Galois group $G = \mathrm{Aut}(\tilde{X}/X)$ such that $\tilde{X}$ is irreducible. Let $K = K(X), L = K(\tilde{X})$ denote their function fields. After assuming X is proper, we have $\mathrm{Aut}(\tilde{X}/X) \cong \mathrm{Gal}(L/K)$, where L/K is an unramified Galois extension. And we will take the identification that $\pi_1^{\text{ét}}(X, \bar{x}) \cong \mathrm{Gal}(K^{ur}/K)$, where K^{ur} is the maximal Galois unramifed extension of K and $\bar{x} \in X$ a geometric point.

Let $x \in |X|$ be a closed point, and consider the local ring $(\mathcal{O}_{X,x}, m_{X,x}, k_x)$. We denote $(\mathcal{O}_{X,x}^h, m_{X,x}, k_x)$ the hensalization of $(\mathcal{O}_{X,x}, m_{X,x}, k_x)$. Let K_x denote the fraction field of $\mathcal{O}_{X,x}^h$. We also have the strict hensalization, denoted by $(\mathcal{O}_{X,x}^{sh}, m_{X,x}, k_x^s)$. We let $K_{\bar{x}}$ denote the fraction field of $\mathcal{O}_{X,x}^{sh}$.

Then we have an embedding of fields

$$K \hookrightarrow K_x \hookrightarrow K_{\bar{x}},$$

whose separable closure satisties

$$K^s \hookrightarrow K_x^s = K_{\bar{x}}^s.$$

It induces an embedding of Galois groups

$$j : \mathrm{Gal}(K_x^s/K_x) \hookrightarrow \mathrm{Gal}(K^s/K).$$

And we can consider the diagram of groups

$$\begin{tikzcd}
\mathrm{Gal}(K_x^s/K_x) \arrow[r] \arrow[d, two heads] & \mathrm{Gal}(K^s/K) \arrow[r, two heads] & \mathrm{Gal}(L/K) \\
\mathrm{Frob} \in \widehat{\mathbb{Z}} \cong \mathrm{Gal}(k_x^s/k_x) & &
\end{tikzcd}$$

Let Frob_x denote the image along the first row of preimage of $\mathrm{Frob} \in \mathrm{Gal}(k_x^s/k_x)$ in $\mathrm{Gal}(K_x^s/K_x)$. It is then well defined up to conjugacy. We then use Frob_x to denote its conjugacy class, sometimes called the Frobenius substitution of $x \in |X|$.

Theorem 3.7 (Chebotarev density theorem)**.** For every conjugacy class $\mathcal{C} \subset G$, the set

$$\{x \in |X|; \mathrm{Frob}_x = \mathcal{C}\}$$

has Dirichlet density

$$\frac{\#\mathcal{C}}{\#G}.$$

Proof. We can prove this theorem by using the idea from the representation theory of finite groups. Let $CF(G)$ denote all the class functions $G \to \mathbb{C}$, i.e.,

$$CF(G) = \{f : G \to \mathbb{C}; f(ghg^{-1}) = f(h), \forall g, h \in G\}.$$

$CF(G)$ has two natural basis: one consists of the characteristic functions of conjugacy classes in G, and the second one consists of the irreducible characters of G.

Let $\varphi_0 \equiv 1, \varphi_1, ..., \varphi_m$ denote the irreducible characters. They correspond to the irreducible representations of G, by sending ρ to $\chi_\rho = \mathrm{trace} \circ \rho$. Moreover, φ_i is an orthonormal basis with the inner product

$$(\varphi, \varphi') = \frac{1}{\#G} \sum_{g \in G} \varphi(g)\overline{\varphi'(g)}.$$

Let $\varphi_{\mathcal{C}}$ be the characteristic function of a conjugacy class $\mathcal{C}$, we have $\varphi_{\mathcal{C}} = \sum_i a_{\mathcal{C},i}\varphi_i$ with

$$a_{\mathcal{C},0} = (\varphi_{\mathcal{C}}, \varphi_0) = \frac{1}{\#G} \sum_{g \in G} \varphi_{\mathcal{C}}(g)\overline{\varphi_0(g)} = \frac{\#\mathcal{C}}{\#G}.$$

Now for any class function φ we can consider the series

$$F_\varphi(s) := \sum_{x \in |X|} \varphi(\mathrm{Frob}_x) \cdot q_x^{-s}.$$

Let $S_{\mathcal{C}} := \{x \in |X|; \mathrm{Frob}_x = \mathcal{C}\}$, then obviously $F_{S_{\mathcal{C}}}(s) = F_{\varphi_{\mathcal{C}}}(s)$. On the other hand, we have $F_{|X|}(s) = F_{\varphi_0}(s)$. Thus we need to prove

$$\lim_{s \to d^+} \frac{F_{\varphi_{\mathcal{C}}}(s)}{F_{\varphi_0}(s)} = (\varphi_{\mathcal{C}}, \varphi_0),$$

for any conjugacy class $\mathcal{C}$. This is equivalent to show

$$\lim_{s \to d^+} \frac{F_{\varphi_i(s)}}{F_{\varphi_0}(s)} = (\varphi_i, \varphi_0) = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i \neq 0 \end{cases},$$

for all i.

When $i = 0$, this is obvious. And by 3.2, it suffices to show that when $i \neq 0$, $F_{\varphi_i}(s)$ is bounded near $s = d$. That is, for any nontrivial irreducible character φ, we want to show $F_\varphi(s)$ is bounded near $s = d$. We now fix such a φ.

As before, we may assume X is geometrically irreducible and $Y = \mathrm{Spec}(\mathbb{F}_q)$. So,

$$F_\varphi(s) = \sum_{n \geq 1} q^{-ns} \sum_{x \in |X|, [k_x : \mathbb{F}_q] = n} \varphi(\mathrm{Frob}_x).$$

We claim that

$$\Delta(s) := F_\varphi(s) - \sum_{n \geq 1} q^{-ns} \frac{1}{n} \sum_{x \in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x)$$

is bounded near $s = d$.

Indeed, any point $x \in |X|$ with $[k_x : \mathbb{F}_q] = n$ corresponds to n primitive points of $X(\mathbb{F}_{q^n})$ as before. Then,

$$F_\varphi(s) - \sum_{n \geq 1} q^{-ns} \frac{1}{n} \sum_{x \in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x) = \sum_{n \geq 1} q^{-ns} \frac{1}{n} \sum_{x \in X(\mathbb{F}_{q^n})^{\mathrm{nonprim}}} \varphi(\mathrm{Frob}_x).$$

So,

$$|\Delta(s)| \leq \sum_{n\geq 1} q^{-n\Re(s)} \frac{1}{n} \sum_{x\in X(\mathbb{F}_{q^n})^{\mathrm{nonprim}}} |\varphi(\mathrm{Frob}_x)| \leq \sum_{n\geq 1} q^{-n\Re(s)} \frac{1}{n} C q^{nd/2} \leq C' |\log(1-q^{\frac{d}{2}-\Re(s)})|,$$

where the second inequality is by (3) in the proof of 3.1 and the fact that G is finite.

As a result, $\Delta(s)$ is bounded near $s = d$.

So, it suffices to bound

$$\sum_{n\geq 1} q^{-ns} \frac{1}{n} \sum_{x\in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x). \tag{*}$$

Choose a number field $E \subset \mathbb{C}$ such that the irreducible representation ρ of G with $\varphi = \mathrm{trace}\circ\rho$ can be defined over E. Let l be a prime such that $l \neq p$ and choose an embedding $E \hookrightarrow \bar{\mathbb{Q}}_l$.

We then consider the composition of maps

$$\pi_1^{\text{ét}}(X, \bar{x}) \twoheadrightarrow G \xrightarrow{\rho} \mathrm{GL}(E) \hookrightarrow \mathrm{GL}(\bar{\mathbb{Q}}_l),$$

which is an l-adic representation and we still denote it by ρ. By our explaination in Section 2, it corresponds to an l-adic $\bar{\mathbb{Q}}_l$-lisse sheaf $\mathcal{F}_l$.

By construction, we know that

$$\varphi(\mathrm{Frob}_x) = \mathrm{trace}(\mathrm{Frob}_x; \mathcal{F}_{l,\bar{x}})$$

for any $x \in |X|$.

By 2.10, for any $n \geq 1$,

$$\sum_{x\in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x) = \sum_{i=0}^{2d} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l)). \tag{**}$$

So,

$$(*) = \sum_{n\geq 1} q^{-ns} \frac{1}{n} \sum_{x\in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x) = \sum_{n\geq 1} q^{-ns} \frac{1}{n} \sum_{i=0}^{2d} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l))$$

$$= \sum_{i=0}^{2d-1} (-1)^i \sum_{n\geq 1} q^{-ns} \frac{1}{n} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l)) + \sum_{n\geq 1} q^{-ns} \frac{1}{n} \mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)).$$

On the other hand, since ρ factors through a finite group G, its eigenvalues is a root of unity. Hence the lisse sheaf is of weight zero, i.e., the complex abslute values of its eigenvalues are one. By 2.14, the eigenvalues of Frob_q on $H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)$ are algebraic numbers of complex absolute value $\leq q^{i/2}$.

So, for the $i \leq 2d-1$ part,

$$|\sum_{i=0}^{2d-1} (-1)^i \sum_{n\geq 1} q^{-ns} \frac{1}{n} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l))|$$

$$\leq \sum_{n\geq 1} q^{-n\Re(s)} \frac{1}{n} \cdot C \cdot q^{n(d-\frac{1}{2})} \leq C \cdot |\log(1-q^{d-1/2-\Re(s)})|,$$

which is bounded near $s = d$.

At last, we only need to bound the $i = 2d$ part

$$\sum_{n\geq 1} q^{-ns} \frac{1}{n} \mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}}, \mathcal{F}_l)).$$

Let $\mathbb{F}_{\tilde{q}}$ be the constant field of $\tilde{X}$ and let $G'' := \mathrm{Gal}(\mathbb{F}_{\tilde{q}}/\mathbb{F}_q)$.

We actually have a commutative diagram with rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1^{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \bar{x}) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[r] \arrow[d, two heads] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G_{geom} \arrow[r] & G \arrow[r] & G'' \arrow[r] & 0
\end{tikzcd}$$

By Poincaré duality 2.16,

$$H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) \cong H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}^\vee(d))^\vee \cong (V_l^\vee(d))^{G_{geom}},$$

where V_l is the fibre of the $\bar{\mathbb{Q}}_l$-sheaf $\mathcal{F}_l$.

Now, let $g \in G_{geom}$ and $v \in (V_l^\vee(d))^{G_{geom}}$, we know $v = g \cdot v = q^d \cdot \rho^\vee(g) \cdot v$. Since $\rho^\vee(g)$ has eigenvalues as roots of unity as before, we know that it must be 1, if $v \neq 0$. So, we know that $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)$ is trivial unless G_{geom} acts on V_l trivially. In this case, ρ factors through G'' and φ comes from an irreducible character φ'' of G''. But since G'' is cyclic, we must know that φ'' has degree 1, i.e., ρ has dimension 1.

Hence $\dim H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) = 1$, with $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) \cong \bar{\mathbb{Q}}_l(d)$ by Poincaré duality as above. Then the eigenvalue of Frob_x is $\varphi''(\mathrm{Frob}_q) \cdot q^d$.

$$\sum_{n\geq 1} q^{-ns} \frac{1}{n} \mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}}, \mathcal{F}_l)) = \sum_{n\geq 1} q^{-ns} \frac{1}{n} (\varphi''(\mathrm{Frob}_q) q^d)^n = -\log(1 - \varphi''(\mathrm{Frob}_q) q^{d-s}). \tag{***}$$

As $\varphi''(\mathrm{Frob}_q)$ is a root of unity and is nontrivial by our assumption on φ, (***) is bounded near $s = d$.

□

The characteristic zero case can be proved similarly by slightly changing the function F_φ.

At last, we want to use the schematic Chebotarev 3.7 to deduce the classic Chebotarev 1.2. Let L/K be a finite Galois extension of number fields and let $X = \mathrm{Spec}(\mathcal{O}_K), Y = \mathrm{Spec}(\mathcal{O}_L)$. $f : Y \to X$ is not an étale cover in general, since it may be ramified at some point. Let $S \subset \mathrm{Spec}(\mathcal{O}_K)$ consists of those primes ramified in the extension L/K. And let $S' \subset \mathrm{Spec}(\mathcal{O}_L)$ be the primes above those in S. Then, the morphism $f : Y \setminus S' \to X \setminus S$ is unramifed, hence étale. The conclusion of 3.7 coincides with that of 1.2.