We are writing this notes because we want to give a review of étale cohomology and a generalization of Chebotarev density theorem for a scheme of finite type over Spec($\mathbb{Z}$).

1 The classic Chebotarev density theorem

1.1 Preliminaries of number theorey

A number field K is a finite extension of $\mathbb{Q}$. We denote its ring of integers $\mathcal{O}_L$, which is the integral closure of $\mathbb{Z}$ in K, and is a Dedekind domain.

Let L/K be a fintie extension of degree n of number fields. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Then the basic commutative algebra tells us that

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_i}...\mathfrak{P}_g^{e_g}$$

for some prime ideals $\mathfrak{P}_j$ of $\mathcal{O}_L$.We call e_j the ramification number of $\mathfrak{P}_j|\mathfrak{p}$.

Let $f_j := [\mathcal{O}_L/\mathfrak{P}_j : \mathcal{O}_K/\mathfrak{p}]$, then we have a equality

$$n = \sum_j e_j f_j$$

by degree chasing.

Moreover, if we let L/K be a Galois extension, then $G := \mathrm{Gal}(L/K)$ acts transitively on the set $\{\mathfrak{P}_1, ..., \mathfrak{P}_g\}$. This implies that if L/K is Galois, the number e_j and f_j are independently of j, and only depend on $\mathfrak{p}$. So, we have

$$n = efg,$$

and

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e}...\mathfrak{P}_g^{e}$$

.

Let $\mathfrak{P}$ be such a prime with $\mathfrak{P}|\mathfrak{p}$. We can define

$$D_{\mathfrak{P}}(L/K) := \{\sigma \in G; \sigma(\mathfrak{P}) = \mathfrak{P}.\},$$

called the decomposition group of $\mathfrak{P}$ in L/K.

In other words, it is the stabilizer group of G acts on the group set $\{\mathfrak{P}_1, ..., \mathfrak{P}_g\}$. As this action is transitive, $D_{\mathfrak{P}}$ has index g, and hence of order ef for any $\mathfrak{P}|\mathfrak{p}$.

Theorem 1.1. Let $k := \mathcal{O}_K/\mathfrak{p}$ and $l := \mathcal{O}_L/\mathfrak{P}$ If L/K is Galois, the so is l/k, and we have a surjection

$$D_{\mathfrak{P}}(L/K) \twoheadrightarrow \mathrm{Gal}(l/k)$$

with kernel $I_{\mathfrak{P}}(L/K)$, called the inertial group.

So, $I_{\mathfrak{P}}$ has order e for any $\mathfrak{P}|\mathfrak{p}$.

Moreover, given an umramified prime $\mathfrak{p}$ of $\mathcal{O}_K$, i.e., a prime $\mathfrak{p}$ with decomposition

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1...\mathfrak{P}_g$$

with $e = 1$.

We then have an isomoprhism of groups

$$D_{\mathfrak{P}}(L/K) \cong \mathrm{Gal}(l/k)$$

for each $\mathfrak{P}|\mathfrak{p}$. Let $\mathrm{Frob}_{\mathfrak{P}}$ denote the Frobenius in $\mathrm{Gal}(l/k)$ and consider it as an element in $D_{\mathfrak{P}}(L/K)$, hence in $\mathrm{Gal}(L/K)$. Since all $D_{\mathfrak{P}}$ are conjugated to each other, this element $\mathrm{Frob}_{\mathfrak{P}}$ is well-defined up to conjugacy. And we shall call it the Froebnius substitution, denoted by $\mathrm{Frob}_{\mathfrak{p}}$ or $(\mathfrak{p}, L/K)$.

1.2 Chebotarev density theorem

The Chebotarev density theorem in algebraic number theory describes statistically the splitting of primes in a given Galois extension K of the field $\mathbb{Q}$ of rational numbers.

Theorem 1.2 (Chebotarev density theorem)**.** Let L be a finite Galois extension of a number field K with Galois group G. Let $\mathcal{C}$ be a conjugacy class in G. Let

$$T := \{\mathfrak{p}; \mathfrak{p} \text{ unramified in } L, (\mathfrak{p}, L/K) = \mathcal{C}\}.$$

Then T has Dirichlet density

$$\delta(T) = \frac{\#\mathcal{C}}{\#G}.$$

We want to explain the classical ways to define all these concepts.

Definition 1.3 (Dirichlet density)**.** Let T be a set of primes of K. If there exists a $\delta(T)$ such that

$$\delta(T) = \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in T}(\#\mathcal{O}_K/\mathfrak{p})^{-s}}{-\log(s-1)},$$

then we call $\delta(T)$ the Dirichlet density of T.

We now want to show the denominator

$$-\log(s-1) \sim \sum_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)} (\#\mathcal{O}_K/\mathfrak{p})^{-s}, s \to 1^+,$$

so that we have the right definition of "density".

At here, let $f(s), g(s)$ be two functions defined for $s > 1$ and real, then we write

$$f(s) \sim g(s), s \to 1^+,$$

if $f(s) - g(s)$ is bounded for

$$1 < s < 1 + \epsilon, s \in \mathbb{R},$$

for some $\epsilon > 0$.

We consider the prime zeta function of the number field K,

$$\zeta_K(s) = \Pi_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)} \frac{1}{1-(\#\mathcal{O}_K/\mathfrak{p})^{-s}}.$$

It's a fact that $\zeta_K(s)$ is meromorphic on a neighborhood of 1 and having a pole of order 1 at 1. So, we can write

$$\zeta_K(s) = \frac{a}{s-1} + g(s), \quad (*)$$

for some a and holomorphic function $g(s)$ in a neighborhood of 1. Moreover, $a > 0$ since $\zeta_K(s) > 0$ for $s > 1$ and real.

Lemma 1.4. Let $u_1, u_2, ...$ be a sequence of real numbers ≥ 2 such that

$$f(s) := \Pi_{j=1}^{\infty} \frac{1}{1-u_j^{-s}}$$

is uniformally conversent on each region

$$D(1,\delta,\epsilon) = \{\Re(s) \geq 1+\delta; |\arg(s-1) \leq \frac{\pi}{2} - \epsilon|\}, \delta, \epsilon > 0.$$

Then

$$\log(f(s)) \sim \sum \frac{1}{u_j^s}, s \to 1^+.$$

Proof. We have

$$\log f(s) = \sum_{j=1}^{\infty} \log \frac{1}{1-u_j^{-s}} = \sum_j \sum_{m=1}^{\infty} \frac{1}{m u_j^{sm}} = \sum_j \frac{1}{u_j^s} + \sum_j \sum_{m=2}^{\infty} \frac{1}{m u_j^{sm}} = \sum_j \frac{1}{u_j^s} + g(s),$$

where

$$|g(s)| \leq \sum_{j=1}^{\infty} \sum_{m=2}^{\infty} |\frac{1}{m u_j^{sm}}| = \sum_{j=1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{m u_j^{\Re(s)m}}.$$

Since $u \geq 2, \Re(s) > 1$,

$$\sum_{m=2}^{\infty} \frac{1}{m u^{m\Re(s)}} \leq \sum_{m=2}^{\infty} \frac{1}{2} (\frac{1}{u^{\Re(s)}})^m = \frac{u^{-2\Re(s)}}{2(1-u^{-\Re(s)})} < \frac{u^{-2\Re(s)}}{1-u^{-2\Re(s)}}.$$

Hence

$$|g(s)| \leq \sum_{j=1}^{\infty} \frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}.$$

Since $f(s)$ is holomorpic in a neighborhood with $\Re(s) > 1$, $f(2s)$ is holomorphic for $\Re(s) > 1/2$, hence bounded near $s = 1$. Since $\sum_{j=1}^{\infty} \frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}$ converges absolutely if and only if $\Pi_j (1 + \frac{u_j^{-2\Re(s)}}{1-u_j^{-2\Re(s)}}) = f(2\Re(s))$ converges absolutely, we know that $g(s)$ is bounded as $s \to 1^+$.

□

As a result, we can take logs to (*), and we find that

$$\sum_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)} (\#\mathcal{O}_K/\mathfrak{p})^{-s} \sim \log(\zeta_K(s)) \sim \log(\frac{1}{s-1}), s \to 1^+,$$

which is what we need.

We want to give an application of the Chebotarev density theorem. We will talk about two extreme spliting types.

Let $\mathrm{Spl}(L/K)$ to denote the primes $\mathfrak{p}$ in $\mathcal{O}_K$ that splits completely in $\mathcal{O}_L$.

Lemma 1.5. $\delta(\mathrm{Spl}(L/K)) = \frac{1}{[L:K]}$

Proof. Let $\mathfrak{p}$ be an unramified prime of $\mathcal{O}_K$. Then $\mathrm{Frob}_{\mathfrak{p}} = 1$ if and only if $\mathfrak{p}$ splits completely. By 1.2, we know

$$\delta(\mathrm{Spl}(L/K)) = \frac{1}{[L:K]}.$$

□

Let $\mathrm{Inert}(L/K)$ be the set of primes $\mathfrak{p}$ in $\mathcal{O}_K$ that is inert, i.e., $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}$ for one prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$.

Then we see that $\mathfrak{p}$ is inert if and only if $\mathrm{Frob}_{\mathfrak{p}}$ generates the whole group G. In other words, if L/K is a cyclic extension, then by 1.2,

$$\delta(\mathrm{Inert}(L/K)) = \frac{\varphi([L:K])}{[L:K]}.$$

And if L/K is not cyclic,

$$\delta(\mathrm{Inert}(L/K)) = 0.$$

2 Review of étale local system and étale cohomology

2.1 étale local system and lisse sheaves

In this section we want to review the Weil Conjecture and Grothendieck-Lefschetz trace formula for vaireties over finite fields, which plays the role as the L-function. We assume the basic knowledge about étale site and étale fundamental group. In particular, given a scheme X over k with a geometric point $x \in X$, we have the short exact sequence

$$0 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, x) \longrightarrow \pi_1^{\text{ét}}(X, x) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 0.$$

Definition 2.1 (Locally constant étale sheaf)**.** Let X be a scheme, and let $\mathcal{F}$ be a sheaf of sets on $X_{\text{ét}}$. We say $\mathcal{F}$ is locally constant if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}_{|U_i}$ is a constant sheaf.

For example, for any prime $l \neq \mathrm{char}(k)$, one can consider the constant sheaf $\underline{\mathbb{Z}/l^n\mathbb{Z}}$ for any $n \geq 1$. And the question is that give an étale covering $\{U_i \to X\}$, how does this glue to a locally constant sheaf, which is fibrewise $\mathbb{Z}/l^n\mathbb{Z}$. This is well-explained by using the étale fundmental group and monodromy theory, which is an analogue to the covering space theory in the manifold case.

Definition 2.2 (Locally constant constructibleétale sheaf)**.** A locally constant constructible étale sheaf is a locally constant étale sheaf, whose fibrewise is a finite set.

From now, we will use $\mathrm{LCC}(X)$ to denote the category of locally constant constructible étale sheaf on X.

Theorem 2.3. Let $X' \to X$ be a finite étale cover of X. Then the functor of points sheaf

$$\underline{X'} := \mathrm{Hom}_X(-, X')$$

on $X_{\text{ét}}$ is a LCC sheaf. Moreover, the association $X' \mapsto \underline{X'}$ gives an equivalence of cateogries

$$\{\text{finite étale covers of } X\} \leftrightarrow \mathrm{LCC}(X)$$

This theorem is prettey strong in the sense that every locally constant sheaf is representable. But this is not too shocking: consider a finite constant sheaf $\underline{\Lambda}$, then it is representable by $\bigsqcup_{x\in\Lambda} X$. And by descent theory, locally representable are representable.

Proof. Let X' be a finite étale cover of X. For any $s \in X$, there exists an open neighborhood (U, x) such that $X' \times_X U \cong \sqcup V_i$, with each $V_i \to U$ isomorphism.

One can show that for étale morphism $\varphi : U \to V$, $\varphi^* h_Z \cong h_{U\times_V Z}$. So, $h_{X'|U} \cong h_{X'\times_X U} = \sqcup \mathrm{Hom}_U(-, V_i) \cong \sqcup \mathrm{Hom}_U(-, U)$, which is constant.

The opposite direction is to use the Étale space of a sheaf, which is a finite étale covering of X. □

Remark: It is not ture in general that the pullback of representable sheaf is representable by the pullback: let X be a k^s-scheme considered as k^s-sheaf. Let $i : k^s \hookrightarrow \bar{k}$ be nontrivial inclusion. Then for any étale $\bar{k}$-algebra L, $i^*h_X(L) = h_X(k^s)^{[L:\bar{k}]}$. So, i^*h_X is constant and not representable by $h_{X_{\bar{k}}}$.

On the other hand, as in the real manifold cases, the finite étale cover of X corresponds to how the fundmental group acting on the fibres. This should be obvious, depending on how the reader realize the étale fundamental group. Indeed, my favorite way is to use the Galois category theory as defining the étale fundamental group as the automorphism group of the fibre functor Fib, see `https://stacks.math.columbia.edu/tag/0BQ6` for more details.

Indeed, given a geometric point $x \in X$, one can define the fibre functor

$$\mathrm{Fib}_x = \varinjlim \mathrm{Hom}_X((U, u), -).$$

The action of $\pi_1^{\text{ét}}(X, x)$ on the finite set $\mathrm{Fib}_x(X')$ is continuous with $\mathrm{Fib}_x(X')$ given the discrete topology, and hence Fib_x gives a natural functor

$$\{\text{finite étale covers of } X\} \longrightarrow \{\text{finite discrete } \pi_1^{\text{ét}}(X, x)\text{-sets}\}.$$

Theorem 2.4 (Grothendieck)**.** Fib_x is an equivalence of categories.

Notice that taking an finite étale cover Y of X, it corresponds to a LCC sheaf $\underline{Y}$ of X. On the other hand, $\mathrm{Fib}_x(Y) = \varinjlim \mathrm{Hom}_X((U, u), (Y, y)) = \varinjlim \underline{Y}(U) = \underline{Y}_{\bar{x}}$ is the stalk. As a result,

$$\mathrm{LCC} \to \{\text{finite discrete } \pi_1^{\text{ét}}(X, x)\text{-sets}\}$$
$$\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$$

gives an equivalence of categories.

Corollary 2.5. Let $X = \mathrm{Spec}(k)$, then we have a equivalence of categoeis

$$\{\text{étale } k\text{-schemes}\} \leftrightarrow \{(\text{finite}) \text{ discrete } \mathrm{Gal}(\bar{k}/k)\text{-sets}\}.$$

Also notice that it preserves group structures, so we also have a equivalence between étale k-group schemes ad the category of finite (discrete) $\mathrm{Gal}(\bar{k}/k)$-groups.

However, one should notice that for the field case, one can do more:

Theorem 2.6. Let k be a field, the one has an equivalence of categories:

$$\mathrm{Sh}(\mathrm{Spec}(k)) \xrightarrow{\sim} \{\text{discrete } \mathrm{Gal}(k^s/k)\text{-sets}\}$$

Proof. Let $\mathcal{F}$ be a k-sheaf. One first notice that for L/k Galois, $\mathcal{F}(K)$ is acted continuously by G. Indeed,

$$\mathcal{F}(K) = \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Spec}(k))}(h_{\mathrm{Spec}(K)}, \mathcal{F})$$

by Yoneda Lemma. As G acts on K continuously, it acts on $h_{\mathrm{Spec}(K)}$, and hence on $\mathcal{F}(K)$. Clearly $\mathcal{F}(K)^{\mathrm{Gal}(K/k)} = \mathcal{F}(k)$. Consider the stalk $\mathcal{F}_{\bar{x}} = \varinjlim \mathcal{F}(L)$ with L/k finite Galois, as it is a cofinal system. It is clear that $\mathcal{F}_{\bar{x}}$ is a discrete G-set.

Alternatively, since the stalk $\mathcal{F}_{\bar{x}}$ is defined to be $\mathcal{F}_{\bar{x}} := \varinjlim_{(U,\bar{u})} \mathcal{F}(U)$ with $(U, \bar{u})$ étale neighborhood of $\bar{x}$. One let $\sigma \in G$ acts on $(U, \bar{u}, s)$ as

$$\sigma \cdot (U, \bar{u}, s) := (U, \sigma(\bar{u}), s).$$

Indeed, we see that $(\mathrm{Spec}(K), \sigma(\bar{u}), s)$ and $(\mathrm{Spec}(K), \bar{u}, \sigma(s))$ defines the same stalk: consider $\mathrm{Spec}(\sigma) : (\mathrm{Spec}(K), (\bar{u})) \to (\mathrm{Spec}(K), \sigma(\bar{u}))$ as a morphism of étale neighborhoods. Then $\mathrm{Spec}(\sigma)^* s = \sigma(s)$.

Conversely, let M be a discrete G-set, then

$$\mathcal{F}_M(A) = \mathrm{Hom}_G(\mathrm{Hom}_k(A, k^s), M)$$

gives a sheaf, where A is étale k-algebra.

Alternatively, since every G-set corresponds to an étale k-scheme X (might not be finite disjoint union of points), one simply define $\mathcal{F}$ to be the sheaf represented by X.

It is easy to verify that these two functors are inverse to each other. □

In particular, one sees that if $\mathcal{F}$ is an k-sheaf, then $\Gamma(\mathcal{F}) = \mathrm{Hom}_G(\{*\}, \mathcal{F}_{\bar{x}}) = \mathcal{F}_{\bar{x}}^G$. It implies any etale sheaf over a field is "locally constant", but this is not true for general schemes obviously. One reason is that we want to use $\mathrm{Gal}(K/k)$ to act on the representable sheaf $h_{\mathrm{Spec}(K)}$, this is okay for fields as this is just automorphism of rings. But for general schemes, $\pi_1^{\text{ét}}(X, \bar{x})$ is defined to be the automorphism group of the fibres $\mathrm{Fib}_{\bar{x}}$ and cannot act on the étale cover Y directly. So, $\pi_1^{\text{ét}}(X, \bar{x})$ cannot act on the sections $\mathcal{F}(U)$ generally.

As an application, $H^i(\mathrm{Spec}(k)_{\text{ét}}, \mathcal{F}) \cong H^i_{cts}(\mathrm{Gal}(k^s/k), \mathcal{F}_{\bar{x}})$ is just the Galois cohomology by the equivalence 2.6 and the equality $\Gamma(\mathcal{F}) = (\mathcal{F}_{\bar{x}})^G$.

Example: Let C be a genus one curve over a field k, it is not always the case that C is an elliptic curve over k since $C(k)$ might be empty. Let $Pic_{C/k}$ be the associated Picard scheme, which is defined to be a group scheme. Let $Pic^0_{C/k} := Jac_{C/k} =: J$ to be the Jacobian variety which is an elliptic curve over k by construction. Then C is a J-torsor. Indeed, étale locally over k, $Pic^0_{C/k} \times Pic^1_{C/k} \to Pic^1_{C/k}$ given by line bundle tensor product gives an action of J on $Pic^1_{C/k}$. And it is well-known that $C \cong Pic^1_{C/k}$. But since $C(k)$ might be empty, there might not be degree-1-k-line bundle in $Pic^1_{C/k}$, and the torsor might nontrivial. The set of isomorphism classes of J-torsors is classified by $H^1_{et}(k, J) \cong H^1(\mathrm{Gal}(k^s/k), J(k^s)) =: WC(J, k)$ is the Weil-Châtelet group of J over k.

Afterall, by 2.3 and 2.4, we can see the locally constant constructible étale sheaf corresponds to the finite $\pi_1^{\text{ét}}(X, x)$-sets S, which corresponds to a continuous representation

$$\pi_1^{\text{ét}}(X, x) \longrightarrow \mathrm{Aut}(S).$$

Also, as $\Gamma(X, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}}(\underline{\mathbb{Z}}, \mathcal{F})$. If $\mathcal{F}$ is locally constant constructible with fibrewise S, then the latter group equals to $\mathrm{Hom}_{\pi_1^{\text{ét}}}(\mathbb{Z}, S)$ by 2.3 and 2.4. It then equals to $S^{\pi_1^{\text{ét}}}$, the invariants of $\pi_1^{\text{ét}}$. Notice that this is exactly the same proof as in the manifold cases.

So far, we only talked about the fintie set S, such as $\mathbb{Z}/l^n\mathbb{Z}$. We want to push this idea further. And we need to introduce the idea of l-adic sheaves.

Definition 2.7 (l-adic sheaf)**.** An l-adic sheaf M is a family $(M_n, f_{n+1} : M_{n+1} \to M_n)$ such that 1. for each n, M_n is a constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules;

2. for each n, the map $f_{n+1} : M_{n+1} \to M_n$ induces an isomoprhism $M_{n+1}/l^n M_{n+1} \to M_n$.

The l-adic sheaf M is said to be locally constant (also called a lisse sheaf)if each M_n is locally constant. So, it corresponds to a continuous representation

$$\rho : \pi_1^{\text{ét}}(X, x) \to \mathrm{Aut}(M).$$

And we can define the cohomology groups

$$H^r_{et}(X, M) = \varprojlim_n H^r_{et}(X, M_n).$$

In particular,

$$H^r_{et}(X, \mathbb{Z}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}).$$

Here is one reason that we have to define the l-adic sheaf in this way is that intuitively, we want to define $H^1(X, \mathbb{Z}_l) = \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}_l)$. But since $\pi_1^{\text{ét}}(X, x)$ has profintie topology, while $\mathbb{Z}_l$ has discrete topology, any continuous homomorphism f must map an open subgroup of $\pi_1^{\text{ét}}$ to 0. Hence f will factor through a finite group, which forces itself to zero.

However, when one consider $\mathbb{Z}_l$ as a lisse sheaf,

$$H^1_{et}(X, \mathbb{Z}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}) = \varprojlim_n \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}/l^n\mathbb{Z}) \cong \mathrm{Hom}_{cts}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}_l),$$

where the last $\mathbb{Z}_l$ carries the l-adic topology.

Similarly, we define a sheaf of $\mathbb{Q}_l$-vector spaces as a lisse sheaf $M = (M_n)$, with

$$H^r_{et}(X, M) = (\varprojlim_n H^r_{et}(X, M_n)) \otimes \mathbb{Q}_l.$$

In particular,

$$H^r_{et}(X, \mathbb{Q}_l) = \varprojlim_n H^r_{et}(X, \mathbb{Z}/l^n\mathbb{Z}) \otimes \mathbb{Q}_l.$$

As a summary, we have that given a $\mathbb{Q}_l$-sheaf $\mathcal{F}$ is the same giving a continuous represetation

$$\rho : \pi_1^{\text{ét}}(X, x) \to \mathrm{GL}(V),$$

where $V = \mathcal{F}_x \cong \mathbb{Q}_l^n$ is the fibre of $\mathcal{F}$ at x.

At last, we give the definition of Tate twist.

Definition 2.8 (Tate twist)**.** Let $\mathbb{Z}_l$ denote the trivial $\mathbb{Z}_l$-sheaf. Then $\mathbb{Z}_l(1)$ is defined as the $\mathbb{Z}_l$-sheaf associated to the lisse sheaf $T_l\mu_{l^\infty} = \varprojlim \mu_{l^n}(k^s) \cong \mathbb{Z}_l^*$ over $\mathrm{Spec}(k)$.

In particular, if $k = \mathbb{F}_q$ is a finite field, then $\mathbb{Q}_l(1)$ associates to the l-adic cyclotomic character

$$\rho : \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to \mathbb{Z}_l^*, \mathrm{Frob}_q \mapsto q.$$

We also define $\mathbb{Z}_l(-1) = \mathbb{Q}_l(1)^\vee$. For any l-adic lisse sheaf $\mathcal{F}$, we define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(1)^{\otimes d}$.

2.2 étale site and topos

We would like to review same facts for the topos on étale sites. Let $f : X \to Y$ be a morphism of schemes, it gives a morphism of sites $f : X_{\text{ét}} \to Y_{\text{ét}}$ given by a functor $u : Y_{\text{ét}} \to X_{\text{ét}}, u(U) = U \times_Y X$, as base change of étale morphism is still étale.

f gives a pair (f^*, f_*), which is a morphism of topoi with $f_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$ and $f^{-1} : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ such that

1. $\mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}(Y_{\text{ét}})}(\mathcal{G}, f_*\mathcal{F})$;
2. f^{-1} commutes with finite limits.

f^{-1} commutes with finite limits is just saying it's exact, since it has a right adjoint. By construction, $f_*\mathcal{F}(U) = \mathcal{F}(U \times_X Y)$. And let $\mathcal{G} \in \mathrm{Sh}(Y_{\text{ét}})$, then we define $f^{-1}\mathcal{G}$ be the sheafification of the presheaf $\mathcal{G}'$ such that for any $f : V \to Y$ étale, $\mathcal{G}'(V) = \varinjlim \mathcal{G}(U)$, with $U \to X$ étale and a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

Definition 2.9 (stalk)**.** Let $\mathcal{F}$ be a (pre)sheaf on the étale site $X_{\text{ét}}$. Let $i : \bar{x} \to X$ be a point of X. The stalk of $\mathcal{F}$ at $\bar{x}$ is defined to be

$$\mathcal{F}_{\bar{x}} := \varinjlim_{(U,\bar{x})} \mathcal{F}(U),$$

where the limit is over the étale neighborhoods on x.

It's clear that $\mathcal{F}_{\bar{x}} = i^{-1}\mathcal{F}(\bar{x})$.

Another useful identification for stalks is the following fact:

Lemma 2.10. Let S bea scheme with $\bar{s} \in S$ a geometric point of S lying over $s \in S$. Let $k = k(s)$ and let $k \subset k^s \subset k(\bar{s})$ denote the separable algebraic closure of k in $k(\bar{s})$. Then there is a canonical identification

$$(\mathcal{O}_{S,s})^{sh} \cong (\mathcal{O}_S)_{\bar{s}},$$

where the left hand side is the strict hensalization of the local ring $\mathcal{O}_{S,s}$ (in the Zariski site), and the right hand site is the stalk of the structure sheaf of $\mathcal{O}_S$ on $S_{\text{ét}}$ at the geometric point $\bar{s}$.

Lemma 2.11. Let

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$$

be a sequence of étale sheaves of abelian groups on $X_{\text{ét}}$, then the followings are equiavlent:

1. the sequence is exact;
2. for any geometric point $\bar{x} \to X$,

$$0 \to \mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

is exact.

We also have a notion for (quasi)-coherent sheaves in the étale site: let X be a scheme and $\mathcal{M}$ a quasi coherent sheaves of X (in the Zariski site). Then it defines a presheaf: for any $\varphi : U \to X$ étale, $\mathcal{M}^{\text{ét}}(\varphi : U \to X) = \Gamma(U, \varphi^*\mathcal{M})$. This actullay gives a sheaf. For convenience, we sometimes use $\widetilde{M}$ to denote the quasi-coherent sheaf on $X_{\text{ét}}$ with respect to M.

Let us give an example that will be used.

Example [Artin-Schreier sequence]:

1. Let R be a commuataive $\mathbb{F}_p$-algebra and consider $\widetilde{R^{perf}}$ as a quasi-coherent sheaf. We claim that the following sequence is exact:

$$0 \to \underline{\mathbb{F}_p} \to \widetilde{R^{perf}} \stackrel{t \mapsto t^p - t}{\longrightarrow} \widetilde{R^{perf}} \to 0.$$

The only nontrivial part is the surjectivity of $t \mapsto t^p - t$. By the above two lemmas, we can reduce to the stalks, which are strict hensalization rings. So, we want to ask if (R, m, k^s) is strict hensal, then does the map $R \to R, t \mapsto t^p - t$ surjective. Let $a \in R$ and consider $t^p - t - a$. $d(t^p - t - a) = -1$, so $t^p - t - a$ is separable and has a simple root in the residue field. Since it is Hensal, it has a root in R, and the surjectivity follows.

2. Similarly, if X is a scheme of characteristic $p > 0$, then we have an exact sequence

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \stackrel{t \mapsto t^p - t}{\longrightarrow} \mathbb{G}_a \to 0.$$

Then it is clear that f^{-1} is exact because for any geometric point $i : \bar{x} \to X$, $(f^{-1}\mathcal{F})_{\bar{x}} = i^{-1}(f^{-1}\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}$. And the exactness follows from stalkwise.

If $f : U \to X$ is an étale morphism, then f^* also has a left adjoint, denoted by $f_!$, called the compactly supported direct image functor or the functor extension by zero.

If $f : U \hookrightarrow X$ is an open immersion, then one can just simply define $f_!\mathcal{F}$ to be the sheafification of the presheaf

$$\mathcal{F}'(V) = \begin{cases} \mathcal{F}(V), \varphi(V) \subseteq U \\ 0, \text{otherwise} \end{cases} \quad , \varphi : V \to X \text{ is étale}$$

Generally, if $f : U \to X$ is étale, then $f_!\mathcal{F}$ is the sheafification of the presheaf

$$\mathcal{F}'(V) = \bigoplus_{\alpha : V \to U \text{ is étale}} \mathcal{F}(V \stackrel{\alpha}{\longrightarrow} U),$$

where $f \circ \alpha = \varphi$ for any $\varphi : V \to X$ étale.

The left adjointness is obvious:

$\mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(f_!\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathbf{PreSh}(X_{\text{ét}})}(\mathcal{F}',\mathcal{G}) = \mathrm{Hom}_{\mathbf{PreSh}(U_{\text{ét}})}(\mathcal{F}, f^{-1}\mathcal{G}) = \mathrm{Hom}_{\mathrm{Sh}(U_{\text{ét}})}(\mathcal{F}, f^{-1}\mathcal{G})$.

Indeed, let $f \in \mathrm{Hom}_{\mathbf{PreSh}(X_{\text{ét}})}(\mathcal{F}',\mathcal{G})$, then for any $\varphi : V \to X$ étale one has a map $f_{|V} : \bigoplus_{\alpha:V\to U \text{ is étale}} \mathcal{F}(V \xrightarrow{\alpha} U) \to \mathcal{G}(V)$. Let $\alpha : V \to U$ be an étale map and $s \in \mathcal{F}(V \xrightarrow{\alpha} U)$. One define $s \mapsto f_{|V}((s_\alpha))_{\alpha:V\to U}$, with $s_\alpha = s, s_{\alpha'} = 0$ for $\alpha' \neq \alpha$. This give a map $f' \in \mathrm{Hom}_{\mathbf{PreSh}(U_{\text{ét}})}(\mathcal{F}, f^{-1}\mathcal{G})$.

Conversely, given a map $f \in \mathrm{Hom}_{\mathbf{PreSh}(U_{\text{ét}})}(\mathcal{F}, f^{-1}\mathcal{G})$. Then for any $\alpha : V \to U$ étale, and $s_\alpha \in \mathcal{F}(V)$, one has a valuation $f(s_\alpha) \in \mathcal{G}(V)$. This naturally give a map $f_{|V} : \bigoplus_{\alpha:V\to U \text{ is étale}} \mathcal{F}(V \xrightarrow{\alpha} U) \to \mathcal{G}(V)$, sending (s_α) to $\sum f(s_\alpha)$.

We also notice that $f_!$ is exact: one first see that for any geometric point $i : \bar{x} \to X$, $(f_!\mathcal{F})_{\bar{x}} \cong \oplus_{\bar{u} \text{ geometric point}, f(\bar{u})=\bar{x}} \mathcal{F}_{\bar{u}}$. As the $\oplus$ only depends on $\bar{x}$, $f_!$ is exact fibrewise.

We explain why $f_!$ is called compactly supported direct image functor.

Lemma 2.12. Let $f : X \to Y$ be a morphism of schdmes which is of finite type. Let $\mathcal{F}$ be an abelian étale sheaf on X. The rule

$$Y_{\text{ét}} \to Ab, V \mapsto \{s \in f_*\mathcal{F}(V) = \mathcal{F}(X_V); \mathrm{Supp}(s) \subset X_V \text{ is proper over } V\}$$

is an abelian subsheaf of $f_*\mathcal{F}$.

Theorem 2.13. Let $j : U \to X$ be a separated étale morphism. Let $\mathcal{F}$ be an abelian sheaf on $U_{\text{ét}}$. Then the image of the injective map $j_!\mathcal{F} \to f_*\mathcal{F}$ is the subsheaf defined in the above lemma.

This theorem implies that if $f : Y \to X$ is finite étale, then $f_! = f_*$.

Definition 2.14 (constructible sheaf)**.** A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$is called constructible if X can be written as a finite (disjoint) union of constructible locally closed subschemes $i_Y : Y \to X$ such that for each subscheme Y,the sheaf $\mathcal{F}|_Y = i_Y^*\mathcal{F}$ is a finite locally constant sheaf.

Note that since closed embedding is not étale generally, a constructible sheaf is not locally constant generally.

Constructible sheaves has a good resolution:

Lemma 2.15. Let X be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form

$$\mathrm{Coker}(\oplus_{j=1,\dots,m} j_{V_j!}\underline{\mathbb{Z}/m_j\mathbb{Z}}_{V_j} \to \oplus_{i=1,\dots,n} j_{U_i!}\underline{\mathbb{Z}/n_i\mathbb{Z}}_{U_i}),$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{\text{ét}}$ and $m_j, n_i > 0$. In fact, we can even assume U_i and V_j afine.

Corollary 2.16. Let $f : A \to B$ be an étale ring homomorphism. Then the functor $f_!$ carries constructible étale sheaves to constructible étale sheaves.

Proof. This follows from the above lemma and the fact that $f_!$ is exact as we proved above. □

Corollary 2.17. Let A be a commutative ring and let $\mathcal{F} \in \mathrm{Sh}^c_{\text{ét}}(\mathrm{Spec}(A), \mathbb{F}_p)$. Then there exists an étale morphism $f : A \to B$ and an epimorhism $f_!\underline{\mathbb{F}_p} \to \mathcal{F}$ in the abelian category $\mathrm{Sh}^c_{\text{ét}}(\mathrm{Spec}(A), \mathbb{F}_p)$.

Proof. By the above lemma, we see there is an epimorphism $\oplus_i j_{U_i!}\underline{\mathbb{F}_p} \to \mathcal{F}$. Let $U := \bigsqcup_i U_i$. Notice that the stalk $(j_{U!}\underline{\mathbb{F}_p})_x \cong \oplus_{u, j_U(u)=x} \mathbb{F}_p \cong \oplus_{U_i} \oplus_{u, j_{U_i!}(u)=x} \mathbb{F}_p \cong (\oplus_i j_{U_i!}\underline{\mathbb{F}_p})_x$. And we are done. □

2.3 Grothendieck-Lefschetz trace formula

In geometry, Lefschetz fixed point theorem is a classical tool to help us check the fixed points of a good map by using the linear data.

Theorem 2.18 (Lefschetz fixed point theorem)**.** Let M be a compact, oriented manifold. Let $f : M \to M$ be a continuous map with isolated fixed points. Then

$$\#\{x \in M : f(x) = x\} = \sum_i (-1)^i \mathrm{trace}(f^*|H^i(M, \mathbb{R})).$$

Inspired by this, Grothendieck has an analogue applied to a variety X over a finite field $k = \mathbb{F}_q$, with $q = p^m$ and $\pi_X := F^m_{X/\mathbb{F}_q} : X \to X$ is the geometric Frobenius.

Definition 2.19 (cohomology with compact support)**.** For any torsion sheaf $\mathcal{F}$ on a variety U, we define

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F}),$$

where X is any complete variety containing U as a dense open subvariety and j is the inclusion map.

Theorem 2.20 (Grothendieck-Lefschetz trace formula)**.** Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf over X. Then

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{trace}(\pi_x|\mathcal{F}_x) = \sum_i (-1)^i \mathrm{trace}(\pi_X^*|H^i_c(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F})),$$

where $\mathcal{F}_x$ is the fibre at x, and π_x is the geometric frobenius at k_x, i.e., the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_q/k_x)$.

In particular, if $\mathcal{F} = \mathbb{Q}_l$ is the constant $\mathbb{Q}_l$-sheaf, the left hand side becomes $\sum_{x \in X(\mathbb{F}_q)} 1 = \#X(\mathbb{F}_q)$. Since $X(\mathbb{F}_q) = \{x \in X(\bar{\mathbb{F}}_q); \pi_X(x) = x\}$, this means we have an analogue and generalization of the Lefschetz fixed point theorem.

Corollary 2.21. Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ a locally constant $\mathbb{Q}_l$-sheaf over X. Then

$$\#X(\mathbb{F}_{q^m}) = \sum_i (-1)^i \mathrm{trace}(\pi_X^m|H^i_c(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_l)).$$

2.4 Weil Conjectures

The trace formula 2.20 inspired Weil that we can use the linear data given by the l-adic cohomology to study the original geometric object. Now, let X be a smooth projective variety of dimension d over $\mathbb{F}_q$.

We can define the zeta function of X by

$$Z(X, t) := \exp(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}) \in \mathbb{Q}[[t]].$$

Alternatively, let $|X|$ be the set of closed points of X. Let $x \in |X|$ be a closed point with residue field k_x. Then

$$Z(X, t) = \Pi_{x \in |X|}(1 - t^{[k_x:\mathbb{F}_q]})^{-1}.$$

One can also define the arithmetic zeta function

$$\zeta_X(t) = \Pi_{x\in|X|}(1 - \#k_x^{-t})^{-1}.$$

Then one can check that $Z(X, q^{-t}) = \zeta_X(t)$.

Note that the arithmetic zeta function is just a generalization of the classic zeta functions for scheme of fintie type over $\mathbb{Z}$. For example, if K is a number field and $\mathcal{O}_K$ its ring of integers, then the Dedekind zeta function is defined as

$$\zeta_K(s) = \Pi_{\mathfrak{p}\subset\mathcal{O}_K}(1 - \#(\mathcal{O}_K/\mathfrak{p}))^{-1}.$$

We want to use the general fact that for an endomorphism φ of a finite dimensional vector space V over a field K, we have an identity of a formal power series

$$\exp(\sum_{n=1}^{\infty} \mathrm{trace}(\varphi^n; V) \cdot \frac{t^n}{n}) = \det(Id - t\cdot\varphi; V)^{-1}. \quad (1)$$

Using 2.21 and (1),

$$Z(X,t) = \exp(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n}) = \exp(\sum_{n=1}\sum_{i}(-1)^i \mathrm{trace}(\pi_X^n|H_c^i(X\times_{\mathbb{F}_q}\bar{\mathbb{F}}_q, \mathbb{Q}_l))\cdot\frac{t^n}{n})$$

$$= \Pi_{i=0}^{2d}\exp(\sum_{n=1}\mathrm{trace}(\pi_X^n; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l))\cdot\frac{t^n}{n})^{(-1)^i} = \Pi_{i=0}^{2d}\det(Id - t\cdot\pi_X; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l))^{(-1)^{i+1}}.$$

Let $P_i(t) := \det(id - t\cdot\pi_X; H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l))$. Suppose it has decomposition over $\mathbb{C}$: $P_i(t) = \Pi_j(1 - \alpha_{ij}t)$. Then α_{ij} are eigenvalues of π_X on $H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$.

Theorem 2.22 (Known as Weil Conjecture, now a theorem of Deligne)**.** Let X be a non-singular projective variety of dimension d over $\mathbb{F}_q$, then

1. (Rationality) The zeta function of V is rational, of the form

$$Z(X,t) = \frac{P_1(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)},$$

where $P_0(t) = 1 - t, P_{2d} = 1 - q^d t$, and each P_i is a polynomial with integer coeffients factoring over $\mathbb{C}$, as $P_i(t) = \Pi_j(1 - \alpha_{ij}t)$.

2. (Functional equation) The zeta function satisfies

$$Z(X, \frac{1}{q^d t}) = \pm q^{\frac{d\cdot\chi(X)}{2}}\cdot t^{\chi(X)}\cdot Z(X,t),$$

where $\chi(X)$ is the Euler chracteristic of X.

3. (Riemann hypothesis) The roorts α_{ij} are algebraic integers of complex absolute value $|\alpha_{ij}| = q^{i/2}$.

By our discussion before the theorem, it means that the eigenvalues of π_X on $H_c^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$ are algebraic integers of complex absolute values $|\alpha_{ij}| = q^{i/2}$.

We give two basic example:

Example 1: Consider the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. Then $H^0(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = H^2(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = \mathbb{Q}_l$ and $H^1(\mathbb{P}^1_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) = 0$. We may take the convention that the determinant of endomorphism on trivial vector space is one. Then

$$Z(\mathbb{P}^1, t) = \frac{1}{(1-t)(1-qt)}.$$

On the other hand, for any n,

$$\#\mathbb{P}^1(\mathbb{F}_{a^n}) = q^n + 1.$$

So, $Z(\mathbb{P}^1, t) = \exp(\sum_{n\geq 1} \frac{q^n+1}{n} t^n)$.

Since $\sum_{n\geq 1} \frac{q^n+1}{n} t^n = -\log(1-qt) - \log(1-t)$, we have the same result.

Example 2: Let X be an abelian varietiey of dimension g over $\mathbb{F}_q$. Then $H^i(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l) \cong \wedge^i H^1(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$. This shows that the eigenvalues α_{ij}'s have the relation that α_{ij} are exactly of the form $\alpha_{ij} = \Pi_{j'\in I, I\subset\{1,\ldots,2g\}, \#I=i} \alpha_{1j'}$. So, the well study of Tate modules of abelian varieties give all information of the zeta functions.

Moreover, Deline later generalized his result to a general lisse sheaf $\mathcal{F}$ of any weight.

Definition 2.23 (weights)**.** Let $n \in \mathbb{Z}$. Let $F_x := \mathrm{Frob}_x^{-1}$.

1. A locally constant sheaf $\mathcal{F}$ of $\mathbb{Q}_l$-vector spaces on X is said to have weight n if for all closed points x of X, each eigenvalue of $F_x : \mathcal{F}_x \to \mathcal{F}_x$ is an algebraic number whose complex conjugates have absolute value $q^{[k_x:\mathbb{F}_q]n/2}$.

2. A fintie-dimensional $\mathbb{Q}_l$-vector space E endowed with a continuous action of $\pi_1^{\text{ét}}(X)$ has weight n, if , for all closed point x of X. each eigenvalue of $\mathrm{Frob}_x : E_x \to E_x$ is an algebraic number whose complex conjugates have absolute value $q^{-[k_x:\mathbb{F}_q]n/2}$.

Example: $\mathbb{Q}_l(1)$ is acted by Frob_q by sending it to q. So, it has weight -2.

Theorem 2.24 (Deligne)**.** Let $f : X \to S$ be a morphism of schemes of finite type over $\mathbb{F}_q$, and let $\mathcal{F}$ be a mixed sheaf of weight $\leq n$ on $X_{\bar{\mathbb{F}}_q}$. Then, for each i, the sheaf $R^i f_! \mathcal{F}$ on S is mixed of weight $\leq n + i$.

In particular, let $S = \mathrm{Spec}(\mathbb{F}_q)$, then for any lisse sheaf $\mathcal{F}$ of weight 0, the eigenvalues of π_X on $H^i_c(X_{\bar{\mathbb{F}}_q}, \mathcal{F})$ are algebraic numbers of complex absolute values $\leq q^{i/2}$.

And he showed it's equality when X is a curve.

Theorem 2.25 (Deigne)**.** Let X be a smooth proper curve over $\mathbb{F}_q$, $j : U \to X$ the inclusion of a dense open subset, and $\mathcal{F}$ a lisse sheaf that is pointwise pure of weight n on U. Then $H^i(X_{\bar{\mathbb{F}}_q}, j^*\mathcal{F})$ is pure of weight $n + i$.

2.5 Poincaré duality

We now want to finish this section by introducing another important and useful tool: the Poincaré duality. Let k be an algebraic closed field and let $\lambda = \mathbb{Z}/n\mathbb{Z}$ for some n prime to the characteristic of k.

We let $\Lambda(1)$ denote μ_n and $\Lambda(m) = \mu_n^{\otimes m}$. Let $\mathcal{F}^\vee(m) = \mathrm{Hom}(\mathcal{F}, \Lambda(m))$ be the hom sheaf.

Theorem 2.26 (Poincaré duality)**.** Let X be a smooth variety of dimension d over an algebraically closed field k.

1. There is a unique map $\eta(X) : H_c^{2d}(X, \Lambda(d)) \to \Lambda$ by sending $cl(P)$ to 1 for any closed point P on X, where $cl(P)$ is the image of 1 under the composition of the Gysin map (which is an isomorphism)

$$H^0(P, \Lambda) \to H_P^{2d}(X, \Lambda(d))$$

and the natural map

$$H_P^{2d}(X, \Lambda(d) \to H_c^{2d}(X, \Lambda(d))).$$

$\eta(X)$ is an isomorphism and called the trace map.

2. For any locally constant sheaf $\mathcal{F}$ of Λ-modules, there are canonical pairings

$$H_c^r(X, \mathcal{F}) \times H^{2d-r}(X, \mathcal{F}^\vee(d)) \to H_c^{2d}(X, \Lambda(d)) \cong \Lambda,$$

which are perfect pairings of finite groups.

As a result, if we let $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$ and take the limits, we have

$$H_c^r(X, \mathcal{F}) \cong H^{2d-r}(X, \mathcal{F}^\vee(d))^\vee$$

for every l-adic lisse sheaf $\mathcal{F}$.

3 Dirichlet density in dimensions ≥ 1

3.1 Dirichlet density

From now we fix a scheme X of finite type over Spec($\mathbb{Z}$). This covers the example of number rings or a variety over a finite field. Let $d := \dim(X)$ and $|X|$ the set of all closed points of X.

If we suppose k is the constant field of X, then $|X|$ equals to the set of all points $x \in X$, with $[k_x : k] \leq \infty$. Moreover, there is a natural bijection

$$|X| \xrightarrow{\cong} X(\bar{k})/\mathrm{Gal}(\bar{k}/k),$$

where the target is the Galois orbits of $\bar{k}$-geometric points in X. In particular, given a closed point $x \in X$ with $[k_x : k] = n$ and k_x/k Galois, it corresponds to a $\mathrm{Gal}(k_x/k)$-orbit $X(k_x)/\mathrm{Gal}(k_x/k)$, which corresponds to exactly n primitive points in $X(k_x)$. At here, a primitive point $x \in X(k_x)$ means a point whose residue field is exactly k_x.

From now, we may suppose $k = \mathbb{F}_q$ for some p-power q. But notice that all theorems of this section apply to characteristic zero cases, with some minor changes in the proofs need to be fixed. We use q_x to denote $\#k_x$ for any closed point $x \in X$. Let Frob_x denote the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_q/k_x)$.

Theorem 3.1. Let $f : X \to Y$ be a proper and smooth morphism of schemes of finite type over Spec$\mathbb{Z}$.

1. Suppose that all fibres of f have dimension $\leq m$, then there exists a constant $C > 0$ such that for all $y \in |Y|$ and all $n \geq 1$, we have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \leq \frac{1}{n} \cdot C \cdot q_y^{nm};$$

2. Suppose that f is surjective and all fibres are geometrically irreducible of dimension $m > 0$. Then there exists a constant $C' > 0$ such that for all $y \in |Y|$ and all $n \geq 1$, we have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \geq \frac{1}{n} \cdot (q_y^{nm} - C' q_y^{n(m-\frac{1}{2})}).$$

Proof. Let X_y denote the fibre of f above y and $k_y^{(n)}$ an extension of k_y of degree n. Then by 2.21, we know

$$\#X_y(k_y^{(n)}) = \sum_{i=0}^{2m} (-1)^i \mathrm{trace}(\mathrm{Frob}_y^n | H_c^i(X_y \times_{k_y} \bar{k}_y), \mathbb{Q}_l).$$

And by 2.22, we know the eigenvalues of Frob_y on $H_c^i(X_y \times_{k_y} \bar{k}_y, \mathbb{Q}_l)$ are algebraic numbers of complex absolute value $\leq q_y^{i/2}$. Moreover, by constructibility and (smooth) proper base change, $R^i f_* \mathbb{Q}_l$ is locally constant and the total number of eigenvalues is bounded independently of y.

So,

$$\#X_y(k_y^{(n)}) = |\#X_y(k_y^{(n)})| \leq C \cdot q_y^{nm} \tag{2}$$

for some constant C.

As we discussed before, any $x \in |X|$ with $f(x) = y$ and $[k_x : \mathbb{F}_q] = n$ corresponds to exactly n primitive points in $X_y(k_y^{(n)})$. This applies (1).

On the other hand, to show the lower bound, we have to give a bound for the non-primitive points in $X_y(k_y^{(n)})$. At here, the non-primitibe points are those in $X_y(k_y^{(n)})$, whose definition field is stirctly contained in $k_y^{(n)}$.

By (2), we have

$$\#(X_y(k_y^{(n)})^{\mathrm{nonprim}}) \leq \sum_{n'|n, m \neq n} \#(X_y(k_y^{(n')})) \leq \sum_{1 \leq n' \leq \frac{n}{2}} C \cdot q_y^{n'm} = C \cdot \frac{q_y^{\lfloor n/2 \rfloor m} - 1}{1 - q_y^{-m}} \leq 2C q_y^{nm/2}. \tag{3}$$

Still by 2.22, we know $H_c^{2m}(X_y \times_{k_y} \bar{k_y}, \mathbb{Q}_l)$ has dimension 1 and the eigenvalue of Frob_y is equal to q_y^m.

We then have

$$\#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} = \frac{1}{n} \cdot \#X_y(k_y^{(n)})^{\mathrm{prim}}$$

$$\geq \frac{1}{n} \cdot (\#(X_y(k_y^{(n)})) - 2C q_y^{nm/2}) \geq \frac{1}{n} \cdot (q_y^{nm} - C q_y^{n(m-1/2)} - 2C q_y^{nm/2})$$

$$\geq \frac{1}{n} \cdot (q_y^{nm} - 3C q_y^{n(m-1/2)}).$$

□

We now want to define the Dirichlet density of points, which is a generalization of number field cases. 3.1 give the well-behaved bounds for us to show the Dirichlet density is well-behaved.

Let X be a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and of dimension $d > 0$. Then for any subset $S \subset |X|$ and a complex parameter s, we can define

$$F_S(s) = \sum_{x \in S} q_x^{-s}. \tag{4}$$

For example, when $X = \text{Spec}\mathbb{Z}$ and $S = |X|$, it's just the prime zeta function

$$F_{|X|}(s) = \sum_p \frac{1}{p^s} \sim \log(\frac{1}{s-1}), s \to 1^+.$$

Lemma 3.2. 1. The series (4) converges absolutely and locally uniformly for $\Re(s) > d$. Thus it deines a holomorphic function in this region.

2. We have

$$\lim_{s \to d^+} F_{|X|}(s) = \infty,$$

where the limit is taken with s approaching d along the real line from the positive direction.

Proof. Let $K = K(X)$ be the function field of X and assume that $\text{char}(K) > 0$. Let $\mathbb{F}_q$ be its field of constants. Note that it's sufficient to replace X by a Zariski dense open subscheme, where one wants to use Noetherian induction for (1).

We may suppose that X is a geometrically irreducible scheme over $Y = \text{Spec}\mathbb{F}_q$. Then we have

$$F_S(s) = \sum_{n \geq 1} q^{-ns} \#\{x \in S; [k_x : \mathbb{F}_q] = n\}.$$

Taking absolute values and using 3.1 (1), we see that this series is dominated by

$$\sum_{n \geq 1} q^{-n\Re(s)} \cdot \frac{1}{n} C q^{nd} = C|\log(1 - q^{d-\Re(s)})|.$$

Clearly this is locally uniformly bounded in the area of $\Re(s) > d$.

For the second conclusion, we take $s \in \mathbb{R}$. By 3.1 (2),

$$F_{|X|}(s) = \sum_{n \geq 1} q^{-ns} \#\{x \in |X|; [k_x : \mathbb{F}_q] = n\} \geq \sum_{n \geq 1} q^{-ns} \frac{1}{n}(q^{nd} - C' q^{n(d-\frac{1}{2})})$$

$$= -\log(1 - q^{d-s}) + C' \log(1 - q^{d-s-\frac{1}{2}}).$$

This goes to infinity clearly.

□

The above Lemma also applies to characteristic zero cases, with Y changed by spectrum of number rings and the sum need to be done over all places of Y.

Definition 3.3 (Dirichlet density)**.** If the limit

$$\mu_X(S) = \lim_{s \to d^+} \frac{F_S(s)}{F_{|X|}(s)}$$

exists, we say that S has a Dirichlet density and $\mu_X(S)$ is called the Dirichlet density of S (in X).

Lemma 3.4. 1. If S has Dirichlet density, then $0 \leq \mu_X(S) \leq 1$.

2. The set $|X|$ has density 1.

3. If S is contained in a Zariski closed proper subset of X, then S has density 0.

4. If $S_1 \subset S \subset S_2 \subset |X|$ such that $\mu_X(S_1)$ and $\mu_X(S_2)$ exist and equals, then $\mu_X(S)$ exists and is equal to $\mu_X(S_1) = \mu_X(S_2)$.

5. For any subsets $S_1, S_2 \subset |X|$, if three of the following densities exist, then so does the fourth and we have

$$\mu_X(S_1 \cup S_2) + \mu_X(S_1 \cap S_2) = \mu_X(S_1) + \mu_X(S_2).$$

Proof. They are easy to see. For (3), S has smaller dimension, so $F_S(s)$ is bounded near $s = d$ by 3.2, while the denominator goes to infinity. □

Lemma 3.5. Let $f : X \to Y$ be a dominant morphism of integral schemes of fintie type over Spec($\mathbb{Z}$). Suppose that $\dim(X) = \dim(Y)$ and that f is totally inseparable at the generic point. Then any given subset $S \subset |X|$ has a density if and only if $f(S)$ has a density, and then $\mu_X(S) = \mu_Y(f(S))$.

Proof. We may choose a Zariski dense open subset $V \subset Y$ such that $U := f^{-1}(U) \to V$ is fintie and totally inseparable at every point. And we may replace X by U and Y by V. Then f induces isomorphisms on the residue fields, since they are all perfect. Hence we have $F_S(s) = F_{f(S)}(s)$ for any subset $S \subset |X|$. □

Theorem 3.6. Let $f : X \to Y$ be a moprhism of schemes of finite type over Spec($\mathbb{Z}$). Suppose that X is integral and that f is non-constant. Then the set

$$\{x \in |X|; [k_x : k_{f(x)}] = 1\}$$

has Dirichlet density 1.

Proof. First we replace Y by the closure of the image of f. Next we write $d = \dim X, e = \dim Y, m = d - e$. Choose a Zariski dense open subset $U \subset X$ such that the fibre dimension $f_{|U} : U \to Y$ is everywhere equal to m. By the lemmas before, we may replace X by U. Now put

$$\{x \in |X|; [k_x : k_{f(x)}] \geq 2\}.$$

By 3.2 (2), we only need to show that $F_S(s)$ converges absolutely and uniformly near $s = d$. Taking absolute values and using 3.1, we see that the series is domonated by

$$\sum_{x \in S} q_x^{-\Re(s)} = \sum_{y \in |Y|} \sum_{x \in S, f(x) = y} q_y^{-[k_x:k_y] \cdot \Re(s)}$$

$$= \sum_{y \in |Y|} \sum_{n \geq 2} q_y^{-n\Re(s)} \cdot \#\{x \in |X|; f(x) = y, [k_x : k_y] = n\} \leq \sum_{y \in |Y|} \sum_{n \geq 2} q_y^{-n\Re(s)} \frac{1}{n} C q_y^{nm}$$

$$\leq \sum_{y \in |Y|} C \frac{q_y^{2(m - \Re(s))}}{1 - q_y^{m - \Re(s)}} \leq \frac{C}{1 - 2^{m - \Re(s)}} \cdot F_{|Y|}(2(\Re(s) - m)).$$

By 3.2 again, this converges locally uniformly for $Re(s) > m + \frac{e}{2} = d - \frac{e}{2}$. Since f is nonconstant, $e > 0$ and this converges uniformly near $s = d$. □

The above theorem is a generalization to the fact that the set of primes of absolute degreee 1 in a number field K has Dirichlet density 1 (in K).

3.2 Chebotarev density theorem

We now can start to state and prove the Chebotarev density theorem. Let X be a scheme of fintie type over Spec($\mathbb{Z}$) as before. Let $\tilde{X} \to X$ be a finite étale Galois covering with Galois group $G = \mathrm{Aut}(\tilde{X}/X)$ such that $\tilde{X}$ is irreducible. Let $K = K(X), L = K(\tilde{X})$ denote their function fields. After assuming X is normal integral, we have $\mathrm{Aut}(\tilde{X}/X) \cong \mathrm{Gal}(L/K)$, where L/K is an unramified Galois extension, i.e., the normalization of X in

L is étale over X. Indeed, if $Y \to X$ is an étale cover with X normal integral, then Y is also normal integral and Y is itself the normalization of X in $K(Y)$.

And we will take the identification that $\pi_1^{\text{ét}}(X, \bar{x}) \cong \text{Gal}(K^{ur}/K)$, where K^{ur} is the maximal Galois unramifed extension of K and $\bar{x} \in X$ a geometric point., i.e., K^{ur} is the union of finite subextensions L of K such that X in unramified in L (the normalization of X in L is étale over X).

Maybe we should give some more details about what we mean, the reference is `https://stacks.math.columbia.edu/tag/0BQJ`. Let X be an integral normal scheme with function field K. Let L/K be a finite extension with Y the normalization of X in L. Then we say X is unramified in L if $Y \to X$ is unramified.

Lemma 3.7. In the situation above the following are equivalent:
1. X is unramified in L;
2. $Y \to X$ is étale;
3. $Y \to X$ is finite étale.

Proof. Omitted. □

If one starts with a finite cover, then the above is automoatically satisfied:

Lemma 3.8. Let X be a normal integral scheme with function field K. Let $Y \to X$ be a finite étale morphism. If Y is connected, then Y is an integral normal scheme and Y is the normalization of X in $K(Y)$.

Proof. Omitted. □

Theorem 3.9. Let X be a normal integral scheme with function field K. The the canonoical map
$$\text{Gal}(K^s/K) \cong \pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \pi_1^{\text{ét}}(X, \bar{\eta})$$
is identification with the quotient map $\text{Gal}(K^s/K) \twoheadrightarrow \text{Gal}(K^{ur}/K)$, with K^{ur} is the union of finite subextensions L of K such that X in unramified in L .

Sketch of proof. We only need to show that given an étale map $f : Y \to X$, $f^{-1}(\eta) = \eta \times_X Y \cong \Pi_i L_i$ with L_i/K separable finite and X unramified in L_i. Without loss of generality, let us assume that $X = \text{Spec}(A)$ is affine with A normal and integral. And $Y = \text{Spec}(B)$ is also affine.

We claim: if A is a normal domain with fraction field K and B an étale A-algebra, then there are finite spearanle field extensions $L_1, ..., L_m$ over K such that $B \otimes_A K \cong \Pi_{i=1}^m L_i$ and $B \cong \Pi_{i=1}^m B_i$, where B_i is the integral closure of A in L_i. Then since B_i is étale over A, A is unramified in L_i by the above lemma.

The claim is easy: $B \otimes_A K = \Pi_{i=1}^m L_i$ as usual and since B/A is flat, we can consider the inculsion $B \hookrightarrow B \otimes_A K$. Since B is finite over A by étaleness, it is integral over A. So, $B \subset \Pi_{i=1}^m B_i$ as defined above. Conversely, B/A is étale implies that B/A is also projective separable, i.e., the A-linear map $\phi : B \to \text{Hom}_A(B, A), \phi(x)(y) = \text{trace}_{B/A}(xy), x, y \in B$ is an isomorphism. Let $x \in \Pi_{i=1}^m B_i$, then since A is integrally closed, we know $\text{trace}_{B \otimes K/K}(xy) \in A$ for any $y \in B$. This map is clearly A-linear, hence $\text{trace}_{B \otimes K/K}(xy) = \text{trace}_{B/A}(x'y)$ for some $x' \in B$ by the separability of B/A. Then $\text{trace}_{B \otimes K/K}(xy) = \text{trace}_{B \otimes K/K}(x'y)$ for all $y \in B \otimes_A K$ by K-linearily. So, $x = x' \in B$. And the claim follows. □

Let $x \in |X|$ be a closed point, and consider the local ring $(\mathcal{O}_{X,x}, m_{X,x}, k_x)$. We denote $(\mathcal{O}^h_{X,x}, m_{X,x}, k_x)$ the hensalization of $(\mathcal{O}_{X,x}, m_{X,x}, k_x)$. Let K_x denote the fraction field of $\mathcal{O}^h_{X,x}$. We also have the strict hensalization, denoted by $(\mathcal{O}^{sh}_{X,x}, m_{X,x}, k^s_x)$. We let $K_{\bar{x}}$ denote the fraction field of $\mathcal{O}^{sh}_{X,x}$.

Then we have an embedding of fields

$$K \hookrightarrow K_x \hookrightarrow K_{\bar{x}},$$

whose separable closure satisties

$$K^s \hookrightarrow K^s_x = K^s_{\bar{x}}.$$

It induces an embedding of Galois groups

$$j : \mathrm{Gal}(K^s_x/K_x) \hookrightarrow \mathrm{Gal}(K^s/K).$$

And we can consider the diagram of groups

$$\begin{array}{ccccc} \mathrm{Gal}(K^s_x/K_x) & \longrightarrow & \mathrm{Gal}(K^s/K) & \twoheadrightarrow & \mathrm{Gal}(L/K) \\ \downarrow & & & & \\ \mathrm{Frob} \in \widehat{\mathbb{Z}} \cong \mathrm{Gal}(k^s_x/k_x) & & & & \end{array}$$

Let Frob_x denote the image along the first row of preimage of $\mathrm{Frob} \in \mathrm{Gal}(k^s_x/k_x)$ in $\mathrm{Gal}(K^s_x/K_x)$. It is then well defined up to conjugacy. We then use Frob_x to denote its conjugacy class, sometimes called the Frobenius substitution of $x \in |X|$.

Let us explain more about the above constructions in different ways. Let (A, m, k) be a normal local ring with fraction field K. Let K^s be the saparable closure of K and A^{sep} be the integral closure of A in K^s. Let m^{sep} be a maximal ideal of A^{sep}. Consider the hensalization and strict hensalization of A, then we have a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & A^h & \longrightarrow & A^{sh} & \longrightarrow & (A^{sep})_{m^{sep}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & K^h & \longrightarrow & K^{sh} & \longrightarrow & K^s \end{array}$$

We have to pass to the hensalization, then the integral closure of a local ring will still be a local ring, and we have a natural identification

$$D = \pi_1^{\text{ét}}(\mathrm{Spec}(K^h)), I = \pi_1^{\text{ét}}(\mathrm{Spec}(K^{sh})), \mathrm{Gal}(k(m^{sep})/k) \cong \pi_1^{\text{ét}}(\mathrm{Spec}(A^h))$$

where $D := \{g \in \mathrm{Gal}(K^s/K); g(m^{sep}) = m^{sep}\}$ is called the decomposition group, and $I := \{g \in D; g = id \mod m^{sep}\}$ is called the inertia group.

Moreover, we have an exact sequence

$$0 \to I \to D \to \mathrm{Gal}(k(m^{sep})/k) \to 0.$$

Note that the decomposition and inertia groups are defined with repsect to the chosen maximal ideals m^{sep}. A differrent chosen will affect those group only up to conjugacy. In other words, it depends on the chosen geometric point $\bar{x}$ for the natural map

$$\pi_1^{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{X,x}^h), \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}).$$

We can also talk about this in finite levels. Indeed, if Y/X is an étale Galois covering with Galois group G. Let us assume Y is the normal closure of X in L, where L is the function field of Y. Let $x \in |X|$ and $y \in |Y|$ above x. Let us also assume every (local) thing is hensal, so the decomposition and inertia groups make sense.

We then have an short exact sequence

$$0 \to I_y(Y/X) \to D_y(Y/X) \to \mathrm{Gal}(k_y/k_x) \to 0,$$

where $D_y(Y/X) := \{g \in G; g(y) = y\}$. Since G acts transitively on the points above x, D_y's are conjugated to each other. And we can see Frob_x is indentified only up to conjugacy. By taking the limit, one then recover the infinite level settings.

Moreover, $I_y(Y/X)$ is actually a trivial group, see (`https://stacks.math.columbia.edu/tag/0BSD`, Lemma 58.13.4).

Another way to see is that given a closed point x, we have a map $x \hookrightarrow X$, which induces a map

$$\pi_1^{\text{ét}}(x, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}).$$

To keep $\pi_1^{\text{ét}}(X, \bar{x})$ identified as the fundamental group of X, this map has to be considered up to conjugacy.

Theorem 3.10 (Chebotarev density theorem)**.** For every conjugacy class $\mathcal{C} \subset G$, the set

$$\{x \in |X|; \mathrm{Frob}_x = \mathcal{C}\}$$

has Dirichlet density

$$\frac{\#\mathcal{C}}{\#G}.$$

Proof. We can prove this theorem by using the idea from the representation theory of finite groups. Let $CF(G)$ denote all the class functions $G \to \mathbb{C}$, i.e.,

$$CF(G) = \{f : G \to \mathbb{C}; f(ghg^{-1}) = f(h), \forall g, h \in G\}.$$

$CF(G)$ has two natural basis: one consists of the characteristic functions of conjugacy classes in G, and the second one consists of the irreducible characters of G.

Let $\varphi_0 \equiv 1, \varphi_1, ..., \varphi_m$ denote the irreducible characters. They correspond to the irreducible representations of G, by sending ρ to $\chi_\rho = \mathrm{trace} \circ \rho$. Moreover, φ_i is an orthonormal basis with the inner product

$$(\varphi, \varphi') = \frac{1}{\#G} \sum_{g \in G} \varphi(g) \overline{\varphi'(g)}.$$

Let $\varphi_{\mathcal{C}}$ be the characteristic function of a conjugacy class $\mathcal{C}$, we have $\varphi_{\mathcal{C}} = \sum_i a_{\mathcal{C},i} \varphi_i$ with

$$a_{\mathcal{C},0} = (\varphi_{\mathcal{C}}, \varphi_0) = \frac{1}{\#G} \sum_{g \in G} \varphi_{\mathcal{C}}(g) \overline{\varphi_0(g)} = \frac{\#\mathcal{C}}{\#G}.$$

Now for any class function φ we can consider the series

$$F_\varphi(s) := \sum_{x\in|X|} \varphi(\mathrm{Frob}_x)\cdot q_x^{-s}.$$

Let $S_\mathcal{C} := \{x \in |X|; \mathrm{Frob}_x = \mathcal{C}\}$, then obviously $F_{S_\mathcal{C}}(s) = F_{\varphi_\mathcal{C}}(s)$. On the other hand, we have $F_{|X|}(s) = F_{\varphi_0}(s)$. Thus we need to prove

$$\lim_{s\to d^+} \frac{F_{\varphi_\mathcal{C}}(s)}{F_{\varphi_0}(s)} = (\varphi_\mathcal{C}, \varphi_0),$$

for any conjugacy class $\mathcal{C}$. This is equivalent to show

$$\lim_{s\to d^+} \frac{F_{\varphi_i(s)}}{F_{\varphi_0}(s)} = (\varphi_i, \varphi_0) = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i \neq 0 \end{cases},$$

for all i.

When $i = 0$, this is obvious. And by 3.2, it suffices to show that when $i \neq 0$, $F_{\varphi_i}(s)$ is bounded near $s = d$. That is, for any nontrivial irreducible character φ, we want to show $F_\varphi(s)$ is bounded near $s = d$. We now fix such a φ.

As before, we may assume X is geometrically irreducible and $Y = \mathrm{Spec}(\mathbb{F}_q)$. So,

$$F_\varphi(s) = \sum_{n\geq1} q^{-ns} \sum_{x\in|X|,[k_x:\mathbb{F}_q]=n} \varphi(\mathrm{Frob}_x).$$

We claim that

$$\Delta(s) := F_\varphi(s) - \sum_{n\geq1} q^{-ns}\frac{1}{n}\sum_{x\in X(\mathbb{F}_{q^n})}\varphi(\mathrm{Frob}_x)$$

is bounded near $s = d$.

Indeed, any point $x \in |X|$ with $[k_x : \mathbb{F}_q] = n$ corresponds to n primitive points of $X(\mathbb{F}_{q^n})$ as before. Then,

$$F_\varphi(s) - \sum_{n\geq1} q^{-ns}\frac{1}{n}\sum_{x\in X(\mathbb{F}_{q^n})}\varphi(\mathrm{Frob}_x) = \sum_{n\geq1} q^{-ns}\frac{1}{n}\sum_{x\in X(\mathbb{F}_{q^n})^{\mathrm{nonprim}}}\varphi(\mathrm{Frob}_x).$$

So,

$$|\Delta(s)| \leq \sum_{n\geq1} q^{-n\Re(s)}\frac{1}{n}\sum_{x\in X(\mathbb{F}_{q^n})^{\mathrm{nonprim}}}|\varphi(\mathrm{Frob}_x)| \leq \sum_{n\geq1} q^{-n\Re(s)}\frac{1}{n}Cq^{nd/2} \leq C'|\log(1-q^{\frac{d}{2}-\Re(s)})|,$$

where the second inequality is by (3) in the proof of 3.1 and the fact that G is finite. As a result, $\Delta(s)$ is bounded near $s = d$.

So, it suffices to bound

$$\sum_{n\geq1} q^{-ns}\frac{1}{n}\sum_{x\in X(\mathbb{F}_{q^n})}\varphi(\mathrm{Frob}_x). \qquad (*)$$

Choose a number field $E \subset \mathbb{C}$ such that the irreducible representation ρ of G with $\varphi = \mathrm{trace} \circ \rho$ can be defined over E. Let l be a prime such that $l \neq p$ and choose an embedding $E \hookrightarrow \bar{\mathbb{Q}}_l$.

We then consider the composition of maps

$$\pi_1^{\text{ét}}(X, \bar{x}) \twoheadrightarrow G \xrightarrow{\rho} \mathrm{GL}(E) \hookrightarrow \mathrm{GL}(\bar{\mathbb{Q}}_l),$$

which is an l-adic representation and we still denote it by ρ. By our explaination in Section 2, it corresponds to an l-adic $\bar{\mathbb{Q}}_l$-lisse sheaf $\mathcal{F}_l$.

By construction, we know that

$$\varphi(\mathrm{Frob}_x) = \mathrm{trace}(\mathrm{Frob}_x; \mathcal{F}_{l,\bar{x}})$$

for any $x \in |X|$.

By 2.20, for any $n \geq 1$,

$$\sum_{x \in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x) = \sum_{i=0}^{2d} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l)). \qquad (**)$$

So,

$$(*) = \sum_{n \geq 1} q^{-ns} \frac{1}{n} \sum_{x \in X(\mathbb{F}_{q^n})} \varphi(\mathrm{Frob}_x) = \sum_{n \geq 1} q^{-ns} \frac{1}{n} \sum_{i=0}^{2d} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l))$$

$$= \sum_{i=0}^{2d-1} (-1)^i \sum_{n \geq 1} q^{-ns} \frac{1}{n} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l)) + \sum_{n \geq 1} q^{-ns} \frac{1}{n} \mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)).$$

On the other hand, since ρ factors through a finite group G, its eigenvalues is a root of unity. Hence the lisse sheaf is of weight zero, i.e., the complex abslute values of its eigenvalues are one. By 2.24, the eigenvalues of Frob_q on $H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)$ are algebraic numbers of complex absolute value $\leq q^{i/2}$.

So, for the $i \leq 2d - 1$ part,

$$\left| \sum_{i=0}^{2d-1} (-1)^i \sum_{n \geq 1} q^{-ns} \frac{1}{n} (-1)^i \mathrm{trace}(\mathrm{Frob}_q^n; H_c^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}_l)) \right|$$

$$\leq \sum_{n \geq 1} q^{-n\Re(s)} \frac{1}{n} \cdot C \cdot q^{n(d-\frac{1}{2})} \leq C \cdot |\log(1 - q^{d-1/2-\Re(s)})|,$$

which is bounded near $s = d$.

At last, we only need to bound the $i = 2d$ part

$$\sum_{n \geq 1} q^{-ns} \frac{1}{n} \mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}}, \mathcal{F}_l)).$$

Let $\mathbb{F}_{\tilde{q}}$ be the constant field of $\tilde{X}$ and let $G'' := \mathrm{Gal}(\mathbb{F}_{\tilde{q}}/\mathbb{F}_q)$.

We actually have a commutative diagram with rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1^{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \bar{x}) & \longrightarrow & \pi_1^{\text{ét}}(X, \bar{x}) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
0 & \longrightarrow & G_{geom} & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 0
\end{array}$$

By Poincaré duality 2.26,

$$H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) \cong H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}^\vee(d))^\vee \cong (V_l^\vee(d))^{G_{geom}},$$

where V_l is the fibre of the $\bar{\mathbb{Q}}_l$-sheaf $\mathcal{F}_l$.

Now, let $g \in G_{geom}$ and $v \in (V_l^\vee(d))^{G_{geom}}$, we know $v = g \cdot v = q^d \cdot \rho^\vee(g) \cdot v$. Since $\rho^\vee(g)$ has eigenvalues as roots of unity as before, we know that it must be 1, if $v \neq 0$. So, we know that $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l)$ is trivial unless G_{geom} acts on V_l trivially. In this case, ρ factors through G'' and φ comes from an irreducible character φ'' of G''. But since G'' is cyclic, we must know that φ'' has degree 1, i.e., ρ has dimension 1.

Hence $\dim H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) = 1$, with $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_l) \cong \bar{\mathbb{Q}}_l(d)$ by Poincaré duality as above. Then the eigenvalue of Frob_x is $\varphi''(\mathrm{Frob}_q) \cdot q^d$.

$$\sum_{n\geq 1} q^{-ns}\frac{1}{n}\mathrm{trace}(\mathrm{Frob}_q^n; H_c^{2d}(X_{\bar{\mathbb{F}}}, \mathcal{F}_l)) = \sum_{n\geq 1} q^{-ns}\frac{1}{n}(\varphi''(\mathrm{Frob}_q)q^d)^n = -\log(1-\varphi''(\mathrm{Frob}_q)q^{d-s}). \tag{***}$$

As $\varphi''(\mathrm{Frob}_q)$ is a root of unity and is nontrivial by our assumption on φ, (***) is bounded near $s = d$.

□

The characteristic zero case can be proved similarly by slightly changing the function F_φ.

At last, we want to use the schematic Chebotarev 3.10 to deduce the classic Chebotarev 1.2. Let L/K be a finite Galois extension of number fields and let $X = \mathrm{Spec}(\mathcal{O}_K), Y = \mathrm{Spec}(\mathcal{O}_L)$. $f : Y \to X$ is not an étale cover in general, since it may be ramified at some point. Let $S \subset \mathrm{Spec}(\mathcal{O}_K)$ consists of those primes ramified in the extension L/K. And let $S' \subset \mathrm{Spec}(\mathcal{O}_L)$ be the primes above those in S. Then, the morphism $f : Y \setminus S' \to X \setminus S$ is unramifed, hence étale. The conclusion of 3.10 coincides with that of 1.2.