In this notes, I want to talk about semistable reductions, semistable representations, Grothendieck's l-adic monodromy theorem and semistable reduction theorem for abelian varieties.

1 Field Theory Setup

Let K be a non-archimedean local field with discrete valuation ring A and residue field κ, where $\mathrm{char}(\kappa) = p$.

Definition 1.1 (Tamely Ramified Extension)**.** We say the finite extension L/K is tamely ramified if it's totally ramified with $(p, e(L/K)) = 1$.

Remember that we have a series of ramification groups of $G = \mathrm{Gal}(L/K)$: $G_{-1} = G \supset G_0 \supset G_1 \supset ...$, where G_0 is the inertia group. And it's known that G_1 is a p-group with G_0/G_1 having order prime to p. So, equivalently, L/K is tamely ramified if and only if $G_1(L/K)$ is trivial.

Consider $\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p$, which is a totally ramified extension having a uniformizer $\varpi = \mu_p - 1$. It is moreover tamely ramified, since $[\mathbb{Q}_p(\mu_p) : \mathbb{Q}_p] = p-1$. Then $\varpi^{p-1} = up \in \mathbb{Q}_p$ for some $u \in \mathbb{Z}_p^*$ and $\mathbb{Q}_p(\mu_p) = \mathbb{Q}_p(p^{1/p-1})$. This is the basic example that for a tamely ramified extension, we can write it by adjoining a prime-to-pth root of the uniformizer.

Theorem 1.1. Consider the maximal unramified extension K^{ur} of K with a uniformizer ϖ. Then L/K^{ur} is tamely unramified if and only if $L = K^{ur}(\varphi^{1/n})$ with $(p, n) = 1$.

Note that this theorem is not valid if K' is not maximal unramified, since we need all roots of unity with prime-to-p orders to make the extension Galois.

Also as a result, the maximal tamely extension field K^{tr} of K^{ur} now has an explicit explanation: $\mathrm{Gal}(K^{tr}/K^{ur}) = \Pi_{l\neq p}\mathbb{Z}_l = \varprojlim_{(n,p=1)} \mu_n(\bar{K})$.

Let $P_K = \mathrm{Gal}(\bar{K}/K^{tr})$ and $I_K = \mathrm{Gal}(\bar{K}/K^{ur})$. Then P_K is the unique maximal pro-p subgroup of I_K and $I_K/P_K \cong \Pi_{l\neq p}\mathbb{Z}_l^*$. P_K is called the wild inertia group of K.

Now we want to talk about the Weil group, which is a subgroup of G_K satisfying the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_K & \hookrightarrow & W_K & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & I_K & \hookrightarrow & G_K & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 0
\end{array}$$

In other words, W_K is the preimage of the discrete subgroup generated by the geometric Frobenius (lift of $\mathrm{Frob}^{-1} : x^p \mapsto x$ in G_K. Notice that the embedding $W_K \hookrightarrow G_K$ is continuous and has dense image. Also, let $\Phi \in G_K$ be a fixed lift of the geometric Frobenius, then $W_K = \bigcup_n I_K \Phi^n$.

Theorem 1.2. There is a canonical isomorphism of topological groups $K^* \xrightarrow{\cong} W_K^{ab}$.

Sketch of proof. :

Let $\nu : K^* \to \mathbb{Z}$ be the normalized valuation on K. Recall that for every $a \in K^*$, we have the "geometric" local reciprocity map $\theta_K(a)_{|K^{ur}} = \mathrm{Frob}_K^{-\nu(a)} \in Gal(K^{ur}/K)$. In particular, we have the "geometric" local reciprocity map $\theta_K : K^* \to \mathrm{Gal}(K^{ab}/K)$ maps K^* into W_K^{ab} and we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_K^* & \longrightarrow & K^* & \xrightarrow{\nu} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \theta_K} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & Gal(K^{ab}/K^{ur}) & \longrightarrow & W_K^{ab} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Now one can argue that $\theta_K : K^* \to W_K^{ab}$ is isomorphism, which we omit here. □

Now we consider the smooth character $||.|| : W_K \to \mathbb{C}^*$ sending I_K to 1 and Φ to q^{-1}, with q the cardinality of κ. Also, it is clear that this coincides with the map

$$W_K \to W_K^{ab} \xrightarrow{\theta_K^{-1}} K^* \xrightarrow{|.|} \mathbb{C}^*,$$

where $|.|$ is normalized so that $|\varpi| = q^{-1}$ for the uniformizer ϖ.

Lemma 1.3. Consider the continuous morphism $t : I_K \to I_K/P_K \cong \Pi_{l \neq p}\mathbb{Z}_l$. Then

$$t(g\tau g^{-1}) = ||g||t(\tau) \text{ for all } \tau \in I_K \text{ and } g \in W_K.$$

Proof. Since W_K is generated by I_K and Φ, it suffices to consider the case $g \in I_K$ and $g = \Phi$. For $g \in I_K$, we want to show $t(g\tau g^{-1}) = t(\tau)$, which is true since $\Pi_{l\neq p}\mathbb{Z}_l$ is abelian.

Now we want to show for the $t(\Phi\tau\Phi^{-1}) = q^{-1}t(\tau)$, or equivalently, $t(\Phi\tau^q\Phi^{-1}) = t(\tau)$. It then suffices to show $\Phi\tau^q \equiv \tau\Phi \mod P_F$. This is equivalently to show $\Phi(\tau^q(\pi^{1/n})) = \tau(\Phi(\pi^{1/n}))$ for any choice $\pi^{1/n}$ with $(n,p) = 1$.

Write $\varpi := \pi^{1/n}$ and $\tau(\varpi) = \mu^m\varpi$ for some n-th primitive root of unity μ and $\Phi(\varpi) = \mu^s\varpi$. Then $\Phi(\tau^q(\varpi)) = \Phi(\mu^{qm}\varpi)) = \mu^{m+s}\varpi$, since $\mu \in K^{ur}$ is fixed by τ. And $\tau(\Phi(\varpi)) = \tau(\mu^s\varpi) = \mu^{m+s}\varpi$. □

Similarly, one can think about $t_l := \mathrm{pr}_l \circ t$, where $\mathrm{pr}_l : \Pi_{l\neq p}\mathbb{Z}_l \twoheadrightarrow \mathbb{Z}_l$. It is also the map $t_l : I_K \to \mathrm{Gal}(K^{tr,l}/K^{ur})$, where $K^{tr,l}$ is obtained to K^{ur} the l^n-th roots for $n \geq 1$ of a uniformizer of K^{ur}. As an analogue to the above lemma, we have $t_l(g^{-1}\tau g) = ||g||t_l(\tau)$ for all $\tau \in I_K$ and $g \in W_K$.

2 Grothendieck's *l*-adic Monodromy Theorem

In this section, we want to prove the Grothendieck's l-adic monodromy theorem:

Theorem 2.1 (Grothendieck's l-adic monodromy theorem)**.** Let $\rho : W_K \to \mathrm{GL}_{\bar{\mathbb{Q}}_l}(V)$ be an l-adic representation, then there is an open subgroup H in the inertia group I_K such that the endomorphism $\rho(h)$ is unipotent for all $h \in H$.

Before we prove this theorem, let us review some basic knowledge about unipotency. Let k be a field with $\mathrm{char}(k) = 0$. Let $A \in M_n(k)$ be a nilpotent matrix. Then $\exp(A) = \sum_{n\geq 0} \frac{A^n}{n!}$ is well defined.

Moreover, if $B \in M_n(k)$ is unipotent, then $\log(B) = -\sum_{n\geq 1} \frac{(I-B)^n}{n}$ is well-defined. And $\exp\circ\log = \log\circ\exp = Id$.

Moreover, we could define $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$ and $\exp(x) = 1 + x + \frac{x^2}{2!} + ...$ for a non-archimedean local field of characteristic 0.

Lemma 2.2. Let E be a non-archimedean local field which is an extension of $\mathbb{Q}_l$ with normalized valuation $\nu : E \twoheadrightarrow \mathbb{Z}$ and $e = \nu(l)$ the ramification index. Then $\log(1+x)$ converges for all $x \in E$ with $\nu(x) > 0$ while $\exp(x)$ converges for all x with $\nu(x) > \frac{e}{l-1}$. Moreover for all $n > \frac{e}{l-1}$, the maps log and exp define mutually inverse isomorphisms of topological groups

$$\log : U_n \to \mathfrak{p}^n, \exp : \mathfrak{p}^n \to U_n.$$

Lemma 2.3. Let G be a profinite group with pro-order prime to l, then every continuous homomorphism $\rho : G \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ has finite image.

Proof. Since G is compact, it suffices to show $\mathrm{Ker}(\rho)$ is open. Consider $\mathcal{O}$ the integral closure of $\mathbb{Z}_l$ in $\bar{\mathbb{Q}}_l$, then $\{K_m = 1 + l^m M_n(\mathcal{O})\}$ gives a descending sequence of open subgroups of $\mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ with the quotient K_m/K_{m+1} abelian group with order a power of l. Let $V := \rho^{-1}(K_1)$. We claim that $V = \mathrm{Ker}(\rho)$. $\mathrm{Ker}(\rho) \subset V$ is obvious. Conversely, V, hence $\rho(V)$ is a profinite group with pro-order prime to l. So, the image of $\rho(V)$ in K_1/K_2 must be trivial and hence $\rho(V) \subset K_2$. By induction, $\rho(V) \subset \bigcap_m K_m = 0$. So, $V \subset \mathrm{Ker}(\rho)$. □

Corollary 2.3.1. Let $\rho : W_K \to \mathrm{GL}_{\bar{\mathbb{Q}}_l}(V)$ be an l-adic representation, then the restriction of ρ to the wild inertia group P_K has finite image.

Proof of Theorem 2.1: By fixing a basis for V we can view ρ as a continuous homomorphism $\rho : W_K \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$. Consider the restriction $\rho_{|\mathrm{Ker}(t_l)} : \mathrm{Ker}(t_l) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$. Since $\mathrm{Ker}(t_l)$ has pro-order prime to l (P_K is pro-p), $\rho_{|\mathrm{Ker}(t_l)}$ has open kernel by the above Lemma. In other words, $\mathrm{Ker}(\rho) \cap \mathrm{Ker}(t_l)$ is an open subgroup of $\mathrm{Ker}(t_l)$. So, $\mathrm{Ker}(\rho) \cap \mathrm{Ker}(t_l) \supset J \cap \mathrm{Ker}(t_l)$ for some open subgroup J of I_K.

Shrinking J if necessary we can assume that $\rho(J) \subset K_2 = 1 + l^2 M_n(\mathcal{O})$. We claim that there is an open normal subgroup H of W_K such that $J \supset H \cap I_K$. Indeed, by the definition of the Krull topology on I_K, we have $J = \mathrm{Gal}(\bar{K}/L)$ for some finite extension L of K^{ur} in $\bar{K}$. Write $L = K^{ur}(\alpha)$ for some $\alpha \in L$. Let $H' = W_K \cap \mathrm{Gal}(\bar{K}/K(\alpha))$, then H' is an open subgroup of W_K with finite index and $H' \cap I_K = J$. Now let H be the intersection of all conjugates of H' in W_K. Then H satisfies the desired conditions.

For simplicity, let $H_1 := H \cap I_K$. Since $H_1 \cap \mathrm{Ker}(t_l) \subset \mathrm{Ker}(\rho)$, the restriction $\rho_{|H_1}$ factors through the homomorphism $H_1 \to t_l(H_1)$. So, there is a homomorphism $\sigma : t_l(H_1) \to K_2$ such that $\rho(h) = \sigma(t_l(h))$ for all $h \in H_1$.

Now $t_l(\Phi h^q \Phi^{-1}) = t_l(h)$ for all $h \in H_1$. Apply σ on both sides we get $\rho(\Phi h^q \Phi^{-1}) = \rho(h)$ for all $h \in H_1$ (remember that H_1 is normal in W_K). In particular, the matrices $\rho(h)$ and $\rho(h^q)$ are conjugate for all $h \in H_1$. So if $\lambda \in \bar{\mathbb{Q}}_l$ is an eigenvalue of $\rho(h)$ then so is λ^q. Since $\rho(h)$ has only finitely many eigenvalues and $\lambda \neq 0$, λ must be a root of unity. On the other hand, since $\rho(H_1) \subset K_2$, $\lambda = 1 + u$ for some $u \in \bar{\mathbb{Q}}_l$ such that u/l^2 is integral over $\mathbb{Z}_l$. In particular, 2.2 says that $\lambda^m = (1+u)^m = 1$ for some $m \geq 1$.

Take the logarithm of both sides one can get $m\log(1+u)=0$. Hence $\log(1+u)=0$ and $\lambda=1+u=\exp(\log(1+u))=1$. Thus, all eigenvalues of $\rho(h)$ are 1 and so $\rho(h)$ is unipotent for all $h\in H_1$.

□

Corollary 2.3.2. With the above notation, there exists a unique nilpotent endomorphism $N: V\to V$ such that for all h in a sufficiently small subgroup of I_K we have $\rho(h)=\exp(t_l(h)N)$.

Proof. Since $\mathrm{Ker}(t_l)$ does not contain any open subgroup of I_K, $t_l(h_0)\neq 0$ for some $h_0\in H_1$. It is clear that the only possible choice for N is

$$N := \tfrac{1}{t_l(h_0)}\log(\rho(h_0)).$$

We want to show that the claim holds for this choice of N. Since $\rho(h_0)$ is unipotent, $\log(\rho(h_0))$ is well defined and nilpotent. Now note that the map $\sigma: t_l(H_1)\to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ in the above proof is continuous. Since two homomorphisms $t_l(H_1)\to\mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ given by $x\mapsto\sigma(x)$ and $x\mapsto\exp(xN)$ agree on $t_l(h_0)$, by continuity they also agree on the closure $\mathbb{Z}_l t_l(h_0)$ of $\mathbb{Z}t_l(h_0)$ in $\mathbb{Z}_l\supset t_l(H_1)$. Set $H:=t_l^{-1}(\mathbb{Z}_l t_l(h_0))$, then H is an open (normal) subgroup of I_K and $\rho(h)=\sigma(t_l(h))=\exp(t_l(h)N)$ for all $h\in H$.

□

It's just a remark here that actually we never used the vector space V is defined over $\bar{\mathbb{Q}}_l$, and one can start with the full Galois group G_K then restrict to W_K:

Theorem 2.4 (Grothendieck's l-adic monodromy theorem)**.** Let $\rho: G_K\to\mathrm{GL}_{\mathbb{Q}_l}(V)$ be an l-adic representation, then there is an open subgroup H in the inertia group I_K such that the endomorphism $\rho(h)$ is unipotent for all $h\in H$.

3 Weil-Deligne Representations

In this section, we give an introduction of the Weil-Deligne representations.

Definition 3.1. Le L be a field of characteristic zero. A Weil-Deligne representation with coefficients in L is a triple (r,V,N) where V is a finite dimensional L-vector space, $r: W_K\to\mathrm{GL}(V)$ is a homomorphism with open kernel (equivalently, r is a continuous morphism with V endowed with the discrete topology) and $N\in\mathrm{End}(V)$ is a nilpotent operator such that $r(g)Nr(g)^{-1}=||g||N$ for all $g\in W_K$.

If (r_1,V_1,N_1) and (r_2,V_2,N_2) are two Weil-Deligne representations over L, a morphism between them is a linear map $f: V_1\to V_2$ such that $f\circ r_1(g)=r_2(g)\circ f$ for all $g\in W_K$ and $f\circ N_1=N_2\circ f$.

Remember $t_l: I_K\to\mathrm{Gal}(K^{tr,l}/K^{ur})\cong\mathbb{Z}_l$ defined by

$$\sigma\mapsto(\tfrac{\sigma(\varpi)}{\varpi})_n;$$

where $\sigma(\varpi)/\varpi$ is a l^n-th root of unity with satisfies the natural compatibility.

Theorem 3.1. Let $L/\mathbb{Q}_l$ be an algebraic extension with $l\neq p$. Fix a lift $\Phi\in W_K$ of the geometric Frobenius and a continuous surjection $t_l: I_K\twoheadrightarrow\mathbb{Z}_l$. Then there is an equivalence of categories

$$\{l\text{-adic representations of } W_K \text{ over } L\} \xrightarrow{\mathrm{WD}_{\Phi,t_l}}$$
$$\{\text{Weil-Deligne representations of } W_K \text{ over } L\}.$$

Moreover, up to a natural isomorphism the functor WD_{Φ,t_l} is independent of the choices of Φ and t_l.

Proof. Let $\rho : W_K \to \mathrm{GL}(V)$ be an l-adic representation with V a finite dimensional vector space over L. By Grothendieck's l-adic monodromy theorem (2.3.2), there is a nilpotent $N \in \mathrm{End}(V)$ such that

$$\rho(h) = \exp(t_l(h)N),$$

for all h in an open normal subgroup of I_K, says I'. Define a map $r : W_K \to \mathrm{GL}(V)$ by

$$r(\Phi^n\sigma) := \rho(\Phi^n\sigma)\exp(-t_l(\sigma)N),$$

for all $n \in \mathbb{Z}$ and $\sigma \in I_K$.

At first, one notice that

We want to show that (r, V, N) is a Weil-Deligne representation.

Claim: $\rho(g)N\rho(g)^{-1} = ||g||$ for all $g \in W_K$.

Proof of claim: $\rho(ghg^{-1}) = \exp(t_l(ghg^{-1})N) = \exp(t_l(h)||g||N)$ for all $g \in W_K, h \in I'$, since I' comes from a normal subgroup in W_K.

On the other hand,

$$\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)^{-1} = \rho(g)\exp(t_l(h)N)\rho(g)^{-1} = \exp(t_l(h)\rho(g)N\rho(g)^{-1}).$$

So, we get the desired identity.

□

Now, for $n_i \in \mathbb{Z}_i, \sigma_i \in I_K$, we have

$$\begin{gathered} r(\Phi^{n_1}\sigma_1\Phi^{n_2}\sigma_2) = \rho(\Phi^{n_1}\sigma_1\Phi^{n_2}\sigma_2)\exp(-t_l(\Phi^{-n_2}\sigma_2\Phi^{n_2}\sigma_2)N) \\ = \rho(\Phi^{n_1}\sigma_1)\rho(\Phi^{n_2}\sigma_2)\exp(-t_l(\Phi^{-n_2}\sigma_1\Phi^{n_2})N)\exp(-t_l(\sigma_2)N) \\ = \rho(\Phi^{n_1}\sigma_1)\rho(\Phi^{n_2}\sigma_2)\exp(-||\Phi||^{-n_2}t_l(\sigma_1)N)\exp(-t_l(\sigma_2)N) \\ = \rho(\Phi^{n_1}N)\exp(-||\Phi^{-n_2}||||\Phi^{n_2}\sigma_2||t_l(\sigma_1)N)\rho(\Phi^{n_2}\sigma_2)\exp(-t_l(\sigma_2)N) \\ = r(\Phi^{n_1}\sigma_1)r(\Phi^{n_2}\sigma_2). \end{gathered}$$

Notice that the fourth equality is by the claim and $||\sigma_2|| = 1$, since $\sigma_2 \in I_K$. And this calculation shows that r is a group homomorphism. Also by construction, r is trivial on I'. So, r is a homomorphism with open kernel.

It remains to show $r(g)Nr(g)^{-1} = ||g||N$ for all $g \in G$. This follows easily from the claim, since

$$\begin{gathered} r(g)Nr(g)^{-1} = \rho(g)\exp(-t_l(\Phi^{-\nu(g)}g)N)N\exp(t_l(\Phi^{-\nu(g)}g)N)\rho(g)^{-1} = \rho(g)N\rho(g)^{-1} \\ = ||g||N. \end{gathered}$$

Now if $f : (\rho_1, V_1) \to (\rho_2, V_2)$ is a morphism, we show that f is also a morphism of Weil-Deligne representations $(r_1, V_1, N_1) \to (r_2, V_2, N_2)$.

Choose $g_0 \in I'$ such that $t_l(g_0) \neq 0$. Then $f \circ N_1 = f \circ \frac{1}{t_l(g_0)}\log(\rho_1(g_0)) = \frac{1}{t_l(g_0)}\log(\rho_2(g_0)) \circ f = N_2 \circ f$.

Now for $g \in G$, we have

$$\begin{gathered} f \circ r_1(g) = f\rho_1(g)\exp(-t_l(\Phi^{-\nu(g)}g)N_1) = \rho_2(g)f\exp(-t_l(\Phi^{-\nu(g)}g)N_1) \\ = \rho_2(g)\exp(-t_l(\Phi^{-\nu(g)})N_2)f = r_2(g) \circ f. \end{gathered}$$

WD_{Φ,t_l} is the a faithful functor.

The functor is obviously essentially surjective: given a Weil-Deligne representation (r, V, N), one can just define $\rho(\Phi^n\sigma) := r(\Phi^n\sigma)\exp(t_l(\sigma)N)$, which gives a continuous l-adic representation $\rho : W_K \to \mathrm{GL}(V)$. And from this, the functor is full.

So, WD_{Φ,t_l} gives an equivalence of categories.

Finally, we show that if Φ' (resp. t'_l) is another choice of Φ (resp. t_l) then the functors WD_{Φ,t_l} and WD_{Φ',t'_l} are naturally isomorphic. We will proceed by showing that $\mathrm{WD}_{\Phi,t_l} \cong \mathrm{WD}_{\Phi',t_l}$ and $\mathrm{WD}_{\Phi,t_l} \cong \mathrm{WD}_{\Phi,t'_l}$.

Indeed, we have $\Phi = \Phi' x$ for some $x \in I_K$. Let (ρ, V) be an l-adic representation and let $(r_1, V, N) := \mathrm{WD}_{\Phi,t_l}((\rho, V))$ and $(r_2, V, N) := \mathrm{WD}_{\Phi',t_l}((\rho, V))$. Note that we have the same nilpotent operator N. Define $f = \exp(\lambda N) \in \mathrm{GL}(V)$, with $\lambda = t_l(x)/(1-q)$. Then $f \circ N = N \circ f$ obviously.

$r_1(\Phi) = \rho(\Phi)$ and $r_2(\Phi) = r_2(\Phi') = r_2(\Phi x) = \rho(\Phi)\exp(-t_l(x)N)$. Want to show:

$$\exp(\lambda N) \circ \rho(\Phi) = \rho(\Phi)\exp(-t_l(x)N)\exp(\lambda N),$$

but this is straight from the claim and our definition for λ.

$r_1(g) = r_2(g)$ for $g \in I_K$, so $f \circ r_1 = r_2 \circ f$ for I_K by the claim and hence for all $\sigma \in W_K$. And then we have $\mathrm{WD}_{\Phi,t_l} \cong \mathrm{WD}_{\Phi',t_l}$.

Now let t'_l be another choice for t_l and so $t'_l = \alpha t_l$ for some $\alpha \in \mathbb{Z}_l^*$. Note that

$$\rho(h) = \exp(t_l(h)N) = \exp(t'_l(h)\alpha^{-1}N),$$

so if $\mathrm{WD}_{\Phi,t_l}((\rho, V)) = (r, V, N)$, then $\mathrm{WD}_{\Phi,t'_l}((\rho, V)) = (r, V, \alpha^{-1}N)$. We therefore need to find $f \in \mathrm{GL}(V)$ such that f commutes with r and $\alpha f \circ N = N \circ f$.

Since $r(\Phi)$ acts on the finite group $r(I_K)$ by conjugation (since $\mathrm{Ker}(r)$ contains a open subgroup I' of I_K), there is some $m \geq 1$ such that $r(\Phi^m)$ commutes with $r(W_K)$. We decompose V as a direct sum of generalized eigenspaces of $r(\Phi)^m$:

$$V = \bigoplus_\lambda \mathrm{Ker}(r(\Phi^m) - \lambda Id_V)^{\dim V}$$

Write $E_\lambda := \mathrm{Ker}(r(\Phi^m) - \lambda Id_V)^{\dim V}$ and define f as a scalar multiplication, i.e., $f_{|E\lambda} = a_\lambda id_{E_\lambda}$ for some $a_\lambda \in L^*$. By our choice of m, each space E_λ is stable under $r(W_K)$ and so f commutes with $r(W_K)$. Furthermore, since $r(\Phi^m)Nr(\Phi^m)^{-1} = ||\Phi^m||N$, it is easy to see that N maps E_λ to $E_{||\Phi^m||\lambda}$. Therefore if we choose the elements $a_\lambda \in L^*$ compatible in the sense that $a_\lambda = \alpha a_{||\Phi^m||\lambda}$ for all λ, then f yields an isomorphism $\mathrm{WD}_{\Phi,t_l} \cong \mathrm{WD}_{\Phi,t'_l}$. □

Lemma 3.2 (Bounded (l-adic) Matrices)**.** Let $A \in \mathrm{GL}_n(L)$ with $L/\mathbb{Q}_l$ is an algebraic extension. Then the following statements are equivalent:

(1) The homomorphism $\mathbb{Z} \to \mathrm{GL}_n(L)$ defined by $1 \mapsto A$ extends to a continuous homomorphism $\widehat{\mathbb{Z}} \to \mathrm{GL}_n(L)$;

(2) A is conjugate to a matrix in $\mathrm{GL}(\mathcal{O}_L)$;

(3) A stabilizes a (maximal) lattice in the L-vector space L^n;

(4) The characteristic polynomial of A has coefficients in $\mathcal{O}_L$ and $\det(A) \in \mathcal{O}_L^*$;

(5) The eigenvalues of A in $\bar{\mathbb{Q}}_l$ are all l-adic units.

Proof. (5)⇔(4) is obvious.

(4)⇒(3): Let J be a lattice in L^n. Define $M := J + AJ + ... + A^{n-1}J$. Then M is also a lattice in L^n and $AM \subset M$ by Cayley-Hamilton and our assumption that the characteristic polynomial of A is in $\mathcal{O}_L[x]$. Moreover, since $\det(A) \in \mathcal{O}_L^*$, A^{-1} also has its characteristic polynomial in $\mathcal{O}_L[x]$ and we therefore can use the same argument to deduce that $A^{-1}M \subset M$, whence $AM = M$.

(3)⇒(2): Clearly since every lattice in L^n has the form $g(\mathcal{O}_L)^{\oplus n}$ for some $g \in GL_n(L)$, and the stabilizer of $(\mathcal{O}_L)^{\oplus n}$ in $\mathrm{GL}_n(L)$ is $\mathrm{GL}_n(\mathcal{O}_L)$.

(2)⇒(4) is clear.

(2)⇒(1): Clearly every homomorphism $\mathbb{Z} \to \mathrm{GL}_n(\mathcal{O}_L)$ extends to a continuous homomorphism $\widehat{\mathbb{Z}} \to \mathrm{GL}_n(\mathcal{O}_L)$ by compactness of $\mathrm{GL}_n(\mathcal{O}_L)$.

(1)⇒(3): The extended homomorphism $\widehat{\mathbb{Z}} \to \mathrm{GL}_n(L)$ yields a continuous action of $\widehat{\mathbb{Z}}$ on the vector space L^n. Since $\widehat{\mathbb{Z}}$ is compact, it can be conjugated into $\mathrm{GL}_n(\mathcal{O}_L)$ and hence stabilizes a lattice in L^n. In particular, A also stabilize a lattice.

□

Definition 3.2. An element $A \in \mathrm{GL}_n(L)$ is called bounded if A satisfies the equivalent conditions in 3.2.

Notice that A is bounded if and only if $A^{\mathbb{Z}}$ in $\mathrm{GL}_n(L)$ is bounded with respect to l-adic topology.

Definition 3.3. Let $r : W_K \to \mathrm{GL}_n(L)$ be a representation with open kernel, then r is bounded if $r(\sigma)$ is bounded for all $\sigma \in W_K$.

Note that since I_K is compact, r is bounded if and only if $r(\sigma)$ is bounded for some $\sigma \in W_K \setminus I_K$.

Theorem 3.3. Fix a choice of Φ and t_l, then the functor WD_{Φ,t_l} restricts to an equivalence of categories:

$$\{l\text{-adic representations of } G_K \text{ over } L\} \overset{\mathrm{WD}_{\Phi,t_l}}{\longrightarrow} \{\text{bounded Weil-Deligne representations of } W_K \text{ over } L\}.$$

Proof. The functor is given by restriction to W_K, so fully faithfulness is obvious, since W_K is dense in G_K. (For any $x \in \mathrm{GL}(V)$, it's centralizer $C_G(x)$ is closed. So if $H \le C_G(x)$, we must have $G \le C_G(x)$, when G is compact and $H \le G$ is dense. In this example, for the given f commuting with W_K, it must commute with I_K and Φ. But G_K/I_K is topologically generated by Φ, so it's obvious).

Now we want to show it's essentially surjective. Let (r, V, N) be the Weil-Deligne representation corresponding to an l-adic representation (ρ, V) of W_K, then

$$\begin{aligned} \rho \text{ extends to } G_K &\Leftrightarrow \mathbb{Z} \to \mathrm{GL}(V), 1 \mapsto \rho(\Phi) \text{ extends to } \widehat{\mathbb{Z}} \\ &\Leftrightarrow \rho(\Phi) \text{ is bounded} \\ &\Leftrightarrow r(\Phi) \text{ is bounded} \\ &\Leftrightarrow r \text{ is bounded.} \end{aligned}$$

So, the functor is essentially surjective and we get the desired equivalence. □

4 Semistable Reduction

In this section, we want to talk reduction theory for abelian varieties. As a beginning, let's recall the well-known criterion for good reductions:

Theorem 4.1 (Néron-Ogg-Shafarevich Criterion)**.** Let A be an abelian variety over a local field K with residue field k. Let l be a prime number different from char(k). Then A has good reduction if and only if the l-adic Tate module $T_l A$ is unramified.

Note that for A to have a good reduction, it means two things: one has an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ and the special fibre of this abelian scheme is smooth.

Remember that for étale group schemes over a hensalian DVR Spec(R) with perfect residue field k, we have an equivalence of categories:

$$\{\text{finite étale } R\text{-group schemes}\} \leftrightarrow \\ \big\{\text{finite abelian groups with a continuous } \mathrm{Gal}(\bar{k}/k)\text{-action}\big\},$$

sending G over R to $G(\bar{k})$. So, any étale group scheme G over R^{ur} must be constant, since its special fibre is. Also $G_{K^{ur}}$ must be constant, which implies that the Galois action on the generic fibre is unramified.

However, one does not have good reduction all the time. But fortunately, the semistable reduction theorem tells us that things are not that bad!

Theorem 4.2 (Semistable Reduction Theorem for Abelian Varieties)**.** Let A/K be an abelian variety over a local field, there exists a finite separable extension K'/K such that $A_{K'}$ has semistable reduction over the integral closure $\mathcal{O}_{K'}$ of $\mathcal{O}_K$ in K'.

We want to remind the readers here what semistable reduction means.

Definition 4.1 (Néron Model)**.** Let R be a Dedekind domain with field of fractions K, and suppose A is a smooth separated scheme over K. Then a Néron model of A_K is defined to be a smooth separated scheme $\mathcal{A}$ over R with generic fibre $\mathcal{A}_K \cong A$ that is universal in the following sense:

(Néron mapping property): If $\mathcal{X}$ is a smooth separated scheme over R then any K-morphism from $\mathcal{X}_K$ to A can be extended to a unique R-morphism from $\mathcal{X}$ to $\mathcal{A}$.

In particular, the canonical map $\mathcal{A}(R) \to A(K)$ is an isomorphism, and if a Néron model exists, it is unique up to a unique isomprhism.

It is known that for an abelian variety A over a local field K, the Néron model $\mathcal{A}$ exists and is a group scheme over R. And if $\mathcal{A}$ is an abelian scheme over R, then $\mathcal{A}$ is the Néron model of $\mathcal{A}_K$. We define $\mathcal{A}_k^0$ to be the identity component of the special fibre of $\mathcal{A}$. Then we say A over K has semistable reduction if $\mathcal{A}_k^0$ is semi-abelian.

Definition 4.2 (semi-abelian variety)**.** A semi-abelian variety G over a field k is a smooth connected commutative k-group G such that G is an extension of an abelian variety by a torus, i.e., we have a short exact sequence over k

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1,$$

with A an abelian variety and T a k-torus.

Notice that by the Chevalley's theorem, it means that $G_{\bar{k}}^{\mathrm{aff}}$ is a torus:

Theorem 4.3 (Chevalley)**.** Let G be a smooth connected group scheme over a perfect field k, then there exists a unique short exact sequence

$$1 \longrightarrow G^{\text{aff}} \longrightarrow G \longrightarrow A \longrightarrow 1,$$

with G^{aff} affine and A an abelian variety.

Proof. Omitted. □

It's worth a remark that in the origin definition for semi-abelian variety, we only require the short exact sequence to be defined over $\bar{k}$. And the current form is a result of the following theorem. Moreover, by the following theorem, we don't need even to ask G to be commutative (although it has to be).

Theorem 4.4. Let G be a smooth connected k-group, and assume that $G^{\text{aff}}_{\bar{k}}$ is a torus. Then G is necessarily commutative and there is a unique short exact sequence of k-groups

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1,$$

for a k-torus T, and abelian variety A over k.

Proof. Omitted. □

Now, let us give a short proof of the Néron-Ogg-Shafarevich criterion, assuming the existence of Néron model.

Sketch of proof of 4.1: If A has good reduction, then it extends to an abelian scheme $\mathcal{A}$, so $\mathcal{A}[l^n]$ is a finite étale group scheme over R, which implies that $T_l(A)$ is unramified.

Conversely, suppose $T_l(A)$ is unramified. Then all the l^n-torsion of A is defined over K^{ur}, and so $\mathcal{A}[l^n](K^{ur})$ has cardinality l^{2ng}, where $g = \dim A$ and $\mathcal{A}$ is the Néron model which will be defined later. Since the Néron model commutes with unramified extension, $\mathcal{A}[l^n](K^{ur}) \cong \mathcal{A}[l^n](R^{ur}) \to \mathcal{A}[l^n](\bar{k})$ is an isomorphism by Hensel's lemma, we see that $\mathcal{A}(\bar{k})[l^n]$ has cardinality l^{2ng}. But $\mathbb{G}_m$ or $\mathbb{G}_a$ has fewer l^n-torsion, which implies that there is no toric or unipotent part of $\mathcal{A}^0_{\bar{k}}$, and so $\mathcal{A}^0_{\bar{k}}$ is an abelian variety, which implies good reduction. □

We will not give a proof of 4.2. Instead, let us now review the reduction theory for elliptic curves over a local field to have some intuitions.

Definition 4.3 (see[4])**.** Let E/K be an elliptic curve and let $\tilde{E}$ be the reduction modulo m_E of a minimal Weierstrass equation of E.

(a) E has good (or stable) reduction if $\tilde{E}$ is smooth;

(b) E has multiplicative (or semistable) reduction if $\tilde{E}$ has a node;

(c) E has additive (or unstable) reduction if $\tilde{E}$ has a cusp.

This semistable definition is related to the previous one by the following theorem:

Theorem 4.5. Let E/K be an elliptic curve given by a minimal Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let Δ be the discriminant of this equation and let c_4 be the usual expression involving $a_1, ..., a_6$. Let $\tilde{E}_{ns}$ be the subgroup involving smooth points.

(a) E has good reduction if and only if $\Delta \in R^*$. In this case $\tilde{E}/k$ is an elliptic curve;

(b) E has multiplicative reduction if and only if $\Delta \in m_K$ and $c_4 \in R^*$. In this case $\tilde{E}_{ns} \cong \mathbb{G}_m$ over $\bar{k}$;

(c) E has additive reduction if and only if $\Delta \in m_E$ and $c_4 \in m_E$. In this case $\tilde{E}_{ns} \cong \mathbb{G}_a$ over $\bar{k}$.

Proof. See [4] Chapter 7, Proposition 5.1. □

As a result, if E is semistable, then $\tilde{E}_{ns} \cong \mathbb{G}_m$ over $\bar{k}$, which means that the Néron model of E has special fibre $\mathbb{G}_m$, which is a semi-abelian variety. So these two definitions agree. Indeed, we have a short exact sequence

$$1 \longmapsto \mathcal{A}_k^{0,\text{aff}} \longrightarrow \mathcal{A}_k^0 \longrightarrow B \longrightarrow 0.$$

If $\dim A = g$, then $\dim \mathcal{A}_k = g$, since a smooth family is automatically flat. So if A is an elliptic curve. We either have $\dim B = 1$, i.e., A has good reduction, or $\dim \mathcal{A}_k^{0,\text{aff}} = 1$. But the only two possible dimension 1 affine groups over $\bar{k}$ are $\mathbb{G}_a$ and $\mathbb{G}_m$ (see [1]).

The next theorem is the semistable reduction theorem and explains the name of stable and semistable.

Theorem 4.6 (Semistable reduction theorem for elliptic curves)**.** Let E/K be an elliptic curve.

(a) Let K'/K be an unramified extension, then the reduction type of E over K (good, multiplicative, or additive) is the same as the reduction type of E over K';

(b) Let K'/K be a finite extension. If E has either good or multiplicative reduction over K, then it has the same reduction type over K';

(c) There exists a finite extension K'/K such that E has either good or multiplicative reduction over K'.

Proof. See [4] Chapter 7, Proposition 5.4. □

5 Grothendieck's inertial semistable reduction criterion

In this section, let's assume the semistable reduction theorem. Let A be an abelian variety over a local field K with residue field k. Let l be a prime number different from char(k). We want to recall the Néron-Ogg-Shafarevich Criterion again.

Theorem 5.1 (Néron-Ogg-Shafarevich Criterion)**.** *A has good reduction if and only if the l-adic Tate module T_lA is unramified.*

Although by the semistable reduction theorem, we know that A always have semistable reduction after a finite field extension. But, we still want to ask is it the case over the smaller field. And Grothendieck gives the criterion, which is not very surprising.

Theorem 5.2 (Grothendieck's inertial semistable reduction criterion)**.** *A has semistable reduction if and only if the I_K-action on V_lA is unipotent.*

It's just a remark that 5.2 plus Grothendieck's l-adic monodromy theorem implies the semistable reduction theorem 4.2 trivially.

The main task in this section is to prove Grothendieck's inertial semistable reduction criterion. The main source is Brain Conrad's notes [3].

Before we start, let us give some preliminaries about group schemes and then show that semi-stability is stable under isogenies.

One thing to know from Chevalley's theorem (4.3) is that it is functorial in the following sense:

Lemma 5.3. If $f : G' \to G$ is a surjective homomorphism (resp. isogeny) between connected k-groups and k is perfect then ${G'}^{\mathrm{aff}} \to G^{\mathrm{aff}}$ is surjective (resp. an isogeny) and likewise for the induced $G'/{G'}^{\mathrm{aff}} \to G/G^{\mathrm{aff}}$ between the abelian parts.

Proof. Omitted. □

And let's remind the readers the structure theorem for smooth connected commutative affine k-group schemes:

Theorem 5.4. Let G be a smooth connected commutative affine k-group.

(1) There is a unique k-torus T in G that contains all k-tori of G, and $U = G/T$ is unipotent;

(2) If k is perfect, then there is a unique splitting $G = T \times U$;

(3) The formation of T and U is functorial in G, and a surjective homomorphism (resp. isogeny) $G' \to G$ between smooth connected commutative affine k-group schemes induces surjective homomorphisms (resp. isogenies) $T' \to T$ and $U' \to U$. In particular, G' is a torus if and only if G is a torus, and like wise for unipotence.

Proof. Omitted. The last claim follows from the fact that there are no non-trivial maps between tori and unipotent groups. □

Corollary 5.4.1. If $f : A \to B$ is an isogeny between group schemes, then either one of A and B is semi-abelian, so is the other.

Proof. f induces an isogeny $f^{\mathrm{aff}} : A^{\mathrm{aff}} \to B^{\mathrm{aff}}$. Without loss of generality, let us assume $k = \bar{k}$, and hence we have an isogeny $f^{\mathrm{aff}} : T_A \times U_A \to T_B \times U_B$.

As a result, $\dim U_A = \dim U_B$. So, if either one of U_A and U_B is trivial, so is the other by considering dimensions. □

Lemma 5.5. Let $f : A \to B$ be a map between abelian varieties over K, and let $\mathcal{A}$ and $\mathcal{B}$ denote the respective Néron models of A and B over R.

(1) If f is an isogeny, and either of $\mathcal{A}_k^0$ or $\mathcal{B}_k^0$ is semi-abelian, the so is the other, and then $f_k^0 : \mathcal{A}_k^0 \to \mathcal{B}_k^0$ is an isogeny;

(2) Conversely, if the reduction f_k^0 is an isogeny, then f is an isogeny.

Proof. There exists an isogeny $h : B \to A$ such that $f \circ h = [n]_B$ and $h \circ f = [n]_A$ for some nonzero integer n. Since multiplication by n is an isogeny on any semi-abelian variety over k, once the equivalence of semi-abelian reductions is established it follows that f_k^0 is an isogeny if and only if h_k^0 is. Hence, by symmetry we may replace f with h if necessary so that $\mathcal{A}_k^0$ is semi-abelian.

Then $\mathrm{Ker}(f_k^0)$ is contained in $\mathrm{Ker}([n]_A)_k^0 = \mathrm{Ker}([n]_{\mathcal{A}_k^0})$, which is finite. Since $\dim \mathcal{A}_k^0 = \dim \mathcal{A}_k = \dim \mathcal{A}_K = \dim A = \dim B = \dim \mathcal{B}_k^0$, f_k^0 is therefore an isogeny. In particular, $\mathcal{B}_k^0$ inherits the semi-abelian property from $\mathcal{A}_k^0$ by 5.4.1.

For (2), the isogeny over k implies equality of dimensions, so $\dim A = \dim B$. Hence, to deduce that f is an isogeny it suffices to prove that it is quasi-finite. Consider the "quasi-finite" locus $U \subset \mathcal{A}$ consists of $a \in \mathcal{A}$ that are isolated in $f^{-1}(f(a))$. This locus is nonempty, since by hypothesis it contains $\mathcal{A}_k^0$. By semi-continuity of fibre dimension for finite type map between Noetherian schemes, U is open. Since $\mathcal{A}$ is R-flat, $\mathcal{A}_K$ is dense in $\mathcal{A}$. It follows that U meets the generic fibre $A = \mathcal{A}_K$. Hence, $f : A \to B$ has an isolated point in some fibre, so by homogeneity f has finite fibres. □

One last step is that we need to invoke the structure theorem for quasi-finite morphisms:

Theorem 5.6. Let X be a quasi-finite and separated over a henselian local ring R. There is a unique decomposition $X = X_f \coprod X_\eta$ where X_f is R-finite and X_η has empty special fibre.

And the formation of the "finite part" X_f is functorial in X and commutes with products, so it is an R-subgroup when X is an R-group.

Proof. Omitted. □

Notice that by the connectedness of $\mathrm{Spec}(R)$, we have $X_\eta(R) = \emptyset$ and thus $X(R) = X_f(R)$.

Now, we fix an abelian variety A of dimension $g > 0$, a DVR R with fraction field K and residue field k of characteristic $p > 0$. We also want to assume A has semistable reduction, i.e., we have a short exact sequence of k-groups:

$$0 \longrightarrow T \longrightarrow \mathcal{A}_k^0 \longrightarrow B \longrightarrow 0,$$

with T a torus, B an abelian variety, and $\mathcal{A}_k^0$ the identity component of the reduction of the Néron model of A.

Suppose $\dim T = t, \dim B = a$, then $g = a + t$. Define $\Phi := \mathcal{A}_k/\mathcal{A}_k^0$, which is a finite étale k-group.

Since the dual abelian variety A' is K-isogenous to A, it also has semistable reduction by 5.5 and a choice of K-isogeny $A' \to A$ induced an isogeny $\mathcal{A}_k'^0 \to \mathcal{A}_k^0$ and thus isogenies $T' \to T$ and $B' \to B$ by the functoriality of Chevalley's theorem. In particular, $\dim T' =: t' = t, \dim B' =: a' = a$.

Fix $N \geq 1$, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \mathcal{A}_k^0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [N]} & & \downarrow{\scriptstyle [N]} & & \downarrow{\scriptstyle [N]} & & \\
0 & \longrightarrow & T & \longrightarrow & \mathcal{A}_k^0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Since T is a torus, $[N] : T \to T$ is an isogeny. By snake lemma, we have

$$0 \to T[N] \to \mathcal{A}_k^0[N] \to B[N] \to 0.$$

As a result, the finite k-group $\mathcal{A}_k^0[N]$ has order N^{t+2a} for $N \geq 1$.

Assume R is henselian, for example, complete DVR, so by the structure theorem for quasi-finite morphisms,

$$\mathcal{A}[N] = \mathcal{A}[N]_f \coprod \mathcal{A}[N]_\eta$$

with $\mathcal{A}[N]_f$ finite over R and having special fibre $\mathcal{A}[N]_k = \mathcal{A}_k[N]$.

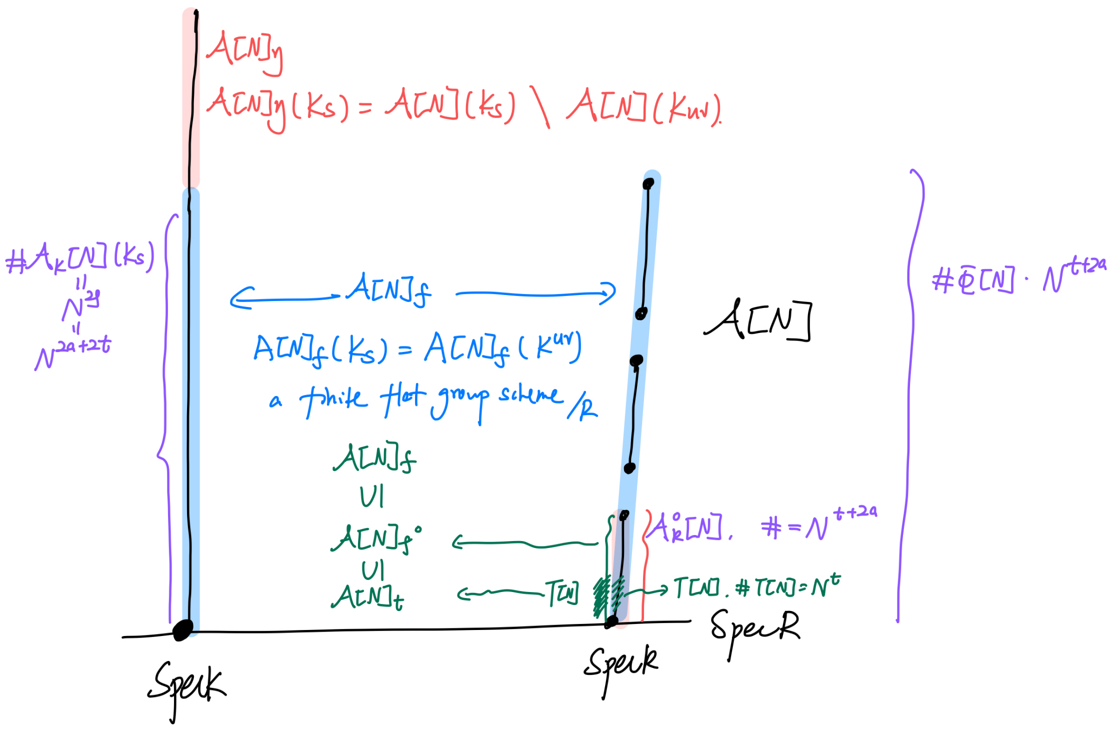
One good thing about semistable is that $[N] : \mathcal{A} \to \mathcal{A}$ is now flat. This is because flatness can be checked fibrewise, and flatness in the special fibre is guaranteed by T! As a corollary, $\mathcal{A}[N]_f$ is also a finite flat R-group.

Consider the unique open finite R-subgroups

$$\mathcal{A}[N]_t \subset \mathcal{A}[N]_f^0 \subset \mathcal{A}[N]_f,$$

which are defined to be the respective lifts of the open and closed k-subgroups $T[N]$ and $\mathcal{A}_k^0[N]$ inside of $\mathcal{A}_k[N]$. The finite flat R-group $\mathcal{A}[N]_t$ has étale Cartier dual, since its special fibre has this property. Take the notations as before, the orders of $\mathcal{A}[N]_t$ and $\mathcal{A}[N]_f^0$ are N^t and N^{t+2a} respectively. One remarks that $\mathcal{A}[N]_t$ is not a torus over R, just having reduction as a torus.

The main reason we need 5.6 is because the quasi-finite group scheme $\mathcal{A}[N]$ over R might not be a finite group scheme. This is because $\mathcal{A}_K[N]$ has rank N^{2g}, while $\mathcal{A}_k[N] = \#\Phi[N] \cdot N^{2a+t}$ with $\Phi[N] \subset \Phi$ bounded. So, we need a finite flat group scheme $\mathcal{A}[N]_f$ to contain all special fibre points and some points in the generic fibre, so that we can move between generic fibre and special fibre in this subgroup scheme over R. Still, this group scheme $\mathcal{A}[N]_f$ has a complicated order related to N, but we need this so that we can play the game with Tate module, since we need finite flat group scheme! The following picture explains this well:



Lemma 5.7. For any prime l, the directed systems $\{\mathcal{A}[l^n]_t\}$ and $\{\mathcal{A}[l^n]_f^0\}$ are l-divisible groups over R of heights t and $t+2a$. Viewing their generic fibres inside of the l-divisible group of A_K, if $l \neq p$ then these yield $\mathrm{Gal}(K_s/K)$-stable saturated $\mathbb{Z}_l$-submodules

$$T_l(A)_t \subset T_l(A)_f \subset T_l A.$$

If $l \neq p$, then $T_l(A)_f = T_l(A)^{I_K}$ is the inertia fixed part.

Proof. We only need to show the last part. We first claim that if $(p, N) = 1$, then the Galois module $\mathcal{A}[N]_f(K_s) \cong \mathcal{A}[N](K^{ur})$ inside $\mathcal{A}[N](K_s)$.

Without loss of generality, let us assume $k = \bar{k}$. We then want to show $\mathcal{A}[N](K) = \mathcal{A}[N]_f(K_s)$. But $\mathcal{A}[N]_f$ is finite constant étale, since $(p, N) = 1$, and then $\mathcal{A}[N]_f(K_s) = \mathcal{A}[N]_f(K)$ (we actually assume R to be hensel, hence the finite étale group scheme over

R is totally controlled by its special fibre, see [2]). By the Néron mapping property, it then suffices to show $\mathcal{A}[N]_f(R) = \mathcal{A}[N](R)$, but this is just how we define $\mathcal{A}[N]_f$.

This actually shows that $\varprojlim \mathcal{A}[N]_f(K_s) = T_l(A)^{I_K}$. However, $\mathcal{A}_k[N]/\mathcal{A}_k^0[N]$ is always contained in $\Phi[N] \subset \Phi$, which is bounded. We know that $\varprojlim \mathcal{A}[N]_f(K_s)/T_l(A)_f$ must be trivial, hence $T_l(A)_f \cong T_l(A)^{I_K}$. □

We call $T_l(A)_t$ the toric part and $T_l(A)_f$ the finite part. So, the toric part has dimension t, while the finite part has codimension $2g - (2a + t) = t$. It suggests that if one study the perfect Weil pairing

$$T_l(A) \times T_l(A') \longrightarrow \mathbb{Z}_l(1),$$

then they are "complement" to each other. This is proved by Grothendieck:

Theorem 5.8 (Grothendieck's orthogonality theorem, semistable case)**.** Under the Weil pairing as above, $T_l(A)_f$ and $T_l(A')_t$ are exact annihilators of each other. In particular, $T_l(A)/T_l(A)_f$ is Cartier dual to $T_l(A')_t$ and hence has trivial I_K-action (as $\mathcal{A}'[N]_t$ has étale Cartier dual for all $N \geq 1$).

As a result, if one replaces K by K^{ur}, and consider the basis of T_lA initialed by $T_l(A)_f$, then $G_K = I_K$ acts trivially on $T_l(A)_f$ and $T_l(A)/T_l(A)_f$. As a result, it must acts unipotently like $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

This proof is inspired by Deligne: By rank consideration, it suffices to show that $T_l(A)_t$ and $T_l(A')_f$ kill each other under the l-adic Weil pairing. The following proof then actually applies to all prime numbers l, even for $l = p$, where one replaces the Tate modules to l-divisible groups.

Let $\Gamma = \left\{\mathcal{A}[l^n]_f^0\right\}$ and $\Gamma' = \{\mathcal{A}'[l^n]_t\}$ be the l-divisible groups over R whose generic fibres respectively define T_lA_f and $T_lA'_t$.

The $\mathbb{Z}_l(1)$-valued pairing between these can be expressed as a homomorphism from one to the Cartier dual of the other. That is, it is encoded in terms of a map of l-divisible groups $f : \Gamma_K \to D(\Gamma')_K$ over K, where $D(\cdot)$ denotes the Cartier duality. By Tate's theorem in his famous paper *p-divisible groups* , we know such a morphism can be uniquely lifted to $f : \Gamma \to D(\Gamma')$ over R, i.e., $\mathrm{Hom}_K(\Gamma_K, D(\Gamma')_K) = \mathrm{Hom}_R(\Gamma, D(\Gamma'))$. We now want to prove that $\mathrm{Hom}_R(\Gamma, D(\Gamma')) = 0$.

Also notice that we have an injection $\mathrm{Hom}_R(\Gamma, D(\Gamma')) \hookrightarrow \mathrm{Hom}_k(\Gamma_k, D(\Gamma')_k)$ for any prime number l. Remember again $\Gamma_k = \{\mathcal{A}_k^0[l^n]_f = \mathcal{A}_k^0[l^n]\}$ So, it suffices to show the following lemma. And we are done.

□

Lemma 5.9 (Grothendieck)**.** Let k be a finite field, G a semi-abelian variety over k and T' a k-torus. For any prime l, $\mathrm{Hom}_k(G[l^\infty], D(T'[l^\infty])) = 0$.

Proof. The strategy of this proof will be called "weight-chasing". Let T be a maximal torus of G and $B = G/T$ the abelian part (over k). We then have an exact sequence of l-divisible groups (by Snake Lemma in the category of commutative group schemes (fppf topology) of finite type over k)

$$1 \longrightarrow T[l^\infty] \longrightarrow G[l^\infty] \longrightarrow B[l^\infty] \longrightarrow 1,$$

so it suffices to prove that any map from the outer terms to $D(T'[l^\infty])$ must vanish.

Making a finite extension of k is harmless, so we may assume T and T' are k-split. Hence, their l-divisible groups are powers of μ_{l^∞}, so we are reduced to proving the vanishing of $\mathrm{Hom}_k(\mu_{l^\infty}, \mathbb{Q}_l/\mathbb{Z}_l)$ and $\mathrm{Hom}_k(B[l^\infty], \mathbb{Q}_l/\mathbb{Z}_l)$.

If $l \neq p$, $\mathrm{Hom}(\mu_{l^\infty}, \mathbb{Q}_l/\mathbb{Z}_l) \cong \mathrm{Hom}_{\mathbb{Z}_l[G_k]}(\mathbb{Z}_l(1), \mathbb{Z}_l) = 0$, since the l-adic cyclotomic character of G_k is nontrivial. If $l = p$, then μ_{l^∞} is an infinitesimal k-group for all n, so obviously $\mathrm{Hom}_k(\mu_{l^\infty}, \mathbb{Q}_l/\mathbb{Z}_l) = 0$.

Now we need to consider $\mathrm{Hom}_k(B[l^\infty], \mathbb{Q}_l/\mathbb{Z}_l)$. First assume $l \neq p$, so we can convert the problem into the language of l-adic Tate modules: does $\mathrm{Hom}_{\mathbb{Z}_l[G_k]}(T_l B, \mathbb{Z}_l)$ vanish? By the Weil Conjecture for abelian varieties over finite fields, the Frobenius action on $T_l B$ does not admit 1 as an eigenvalue. So $\mathrm{Hom}_{\mathbb{Z}_l[G_k]}(T_l B, \mathbb{Z}_l) = 0$. If $l = p$, we convert the problem into the language of contravariant Dieudonné modules instead: we need to show there are no nonzero $W(k)$-linear maps from $W(k)$ to the Dieudonné module of $B[p^\infty]$ that are compatible with the action of the absolute Frobenius operators, or the $W(k)$-linear q-Frobenius operators. But the q-Frobenius operators act on the Dieudonné module of B also cannot have eigenvalue 1, since the characteristic polynomial comes from the characteristic polynomial associated to the endomorphism π_B. □

Now we are ready to prove Grothendieck's inertial semistable reduction criterion. One remember that each $g \in I_K$ acts as a unipotent automorphism (the pointwise sense) is equivalent to the action of I_K can be strictly upper-triangulized (the group-theoretic sense).

Proof of 5.2: Suppose A has semistable reduction, then the claim follows from our calculation after 5.8, which is a strict corollary of the Grothendieck's orthogonality theorem.

For the converse, let us replace R with its strict henselization R^{sh}. So, without loss of generality, we assume $k = \bar{k}$ and $G_K = I_K$ has unipotent action.

By the semistable reduction theorem 4.2, there is a finite separable extension K'/K inside of K_s such that the Néron model $\mathcal{A}'$ of $A_{K'}$ over the local integral closure R' of R in K' has special fibre with semi-abelian identity component. Thus, we have a short exact sequence

$$0 \longrightarrow T' \longrightarrow \mathcal{A}'^0_k \longrightarrow B' \longrightarrow 0$$

of group varieties over k, where T' is a torus and B' an abelian variety. We now want to descent these.

We claim that he inclusion $V_l(A)^{G_K} \subset V_l(A)^{G_{K'}}$ is an equality.

proof of claim: For any $g \in G_K$, some power g^n lies in $G_{K'}$ with $n > 0$. Thus, it suffices to show that if g is a unipotent automorphism of a finite dimensional vector space over a field of characteristic zero, then $g - 1$ and $g^n - 1$ have the same kernel.

Write $g = 1 + N$ for some nilpotent N, we have

$$g^n = 1 + nN + N^2(...) = 1 + nN(1 + N(...)).$$

Thus $g^n - 1 = nN(1 + N(...))$, where $1 + N(...)$ is invertible. Since n acts invertible (here we use the field has characteristic zero), we are done. □

The claim implies that $A[l^n](K) = A[l^n](K')$ for all $n \geq 1$, so by the Néron mapping property, we have $\mathcal{A}[l^n](R) = \mathcal{A}'[l^n](R')$ for all $n \geq 1$. Since $l \neq p$, and $k = \bar{k}$, all finite flat commutative group scheme over R' of l-power order are constant. Also, even though $\mathcal{A}^0_k$ is not known to be semi-abelian, the endomorphism of multiplication by l is étale, so

$l : \mathcal{A} \to \mathcal{A}$ over R is étale. Hence, all $\mathcal{A}[l^n]$'s are quasi-finite étale over R and thus $\mathcal{A}[l^n]_f$ makes sense as a finite constant R-subgroup of $\mathcal{A}$.

It follows that $\mathcal{A}'[l^n]_t$ and $\mathcal{A}'[l^n]_f^0$ (viewed as finite constant R'-groups) descent into the finite constant R-group $\mathcal{A}[l^n]_f$, as all R-points of $\mathcal{A}[l^n]$ factor through the finite part. Passing to the special fibre, inside of the mysterious $\mathcal{A}_k$ we have produced two constant l-divisible subgroups $\Gamma_t \subset \Gamma$ such that the special fibre $\mathcal{A}_k \to \mathcal{A}'_k$ of the base change morphism carries Γ_t isomorphically onto $T'[l^\infty]$ and Γ isomorphically onto $\mathcal{A}'^0_k[l^\infty]$.

Let $T \subset \mathcal{A}^0_k$ denote the identity component of the Zariski closure in $\mathcal{A}_k$ of the points in $\Gamma_t(k) \subset \mathcal{A}_k(k)$. By the density reason, the map $\mathcal{A}_k \to \mathcal{A}'_k$ carries T into T'. But the image of T in T' contains $T'[l^\infty]$, so T_k maps onto T' ($\mu_{l^\infty} \subset \mathbb{G}_m$ is dense). It follows that $\dim T \geq \dim T'$, with equality if and only if $T \twoheadrightarrow T'$ is an isogeny, in which case T must be a torus by 5.4.

Consider the constant l-divisible group Γ in $\mathcal{A}_k$. By finiteness of the component group of $\mathcal{A}_k$, Γ is contained in $\mathcal{A}^0_k$. Under the special fibre $f : \mathcal{A}^0_k \to \mathcal{A}'^0_k$ of the base change morphism, Γ is carried isomorphically onto $\mathcal{A}'^0_k[l^\infty]$, which is Zariski-dense in the semi-abelian $\mathcal{A}'^0_k$. Hence, f is surjective. But we have seen that $f(T) = T'$, so the induced map $\bar{f} : \mathcal{A}^0_k/T \to \mathcal{A}'^0_k/T'$ is surjective, with target an abelian variety.

Since $\dim(\mathcal{A}^0_k/T) = \dim \mathcal{A}^0_k - \dim T = g - \dim T \leq g - \dim T' = \dim \mathcal{A}'^0_k/T'$, it follows that $\bar{f}$ must be an isogeny and $\dim T = \dim T'$. We conclude that T is a torus and $\mathcal{A}^0_k/T$ is an abelian variety.

□

6 Semistable Representations

In this section, we travel to p-adic case and consider the semistable representations in the p-adic Hodge theory sense. For those who want to learn more about the integral p-adic Hodge theory, one can go to read my notes on that. The goal of this idea is to give an analogue of semistable reduction in the p-adic setting, which is very different from l-adic setting. Informally, a representation (from geometry) has good reduction if it's crystalline and it has semistable reduction if it's semistable. However, there is a hard theorem saying that every de Rham representation is potentially semistable.

Definition 6.1 ((ϕ, N)-module)**.** A (ϕ, N)-module (over K_0) is an isocrystal (D, ϕ_D) over K_0 equipped with a K_0-linear endomorphism $N_D : D \to D$ (called the monodromy operator) such that $N_D\phi_D = p\phi_D N_D$. The notion of morphism between such objects is the evident one. The category of these is denoted $\mathrm{Mod}^{\phi,N}_{K_0}$.

A filtered (ϕ, N)-module (over K) is a (ϕ, N)-module D over K_0 for which D_K is endowed with a structure of object in Fil_K. The notion of morphism between such objects is the evident one, and the category of these is denoted $\mathrm{MF}^{\phi,N}_K$.

Notice that we have the natural induced monodromy operators:

$$N_{D\otimes D'} = Id_D \otimes N_{D'} + N_D \otimes ID_{D'} \text{ and } N_{D^\vee} = -N_D^\vee.$$

These are analogues from the l-adic setting by checking $\rho(g) = \exp(t_l(g)N)$.

Lemma 6.1. For any $D \in \mathrm{Mod}^{\phi,N}_{K_0}$, the monodromy operator N_D on D is nilpotent. In particular, if $\dim D = 1$ then $N_D = 0$.

Proof. Consider the isoclinic decomposition $D = \bigoplus_{\alpha\in\mathbb{Q}} D(\alpha)$ of the underlying isocrystal. By the definition of $D(\alpha)$, its scalar extension $D'(\alpha) := D(\alpha)_{\widehat{K_0^{ur}}}$ is spanned by vectors such that $\phi^r_{D'}(v) = p^s v$ for $s/r = \alpha$, so

$$\phi^r_{D'}(Nv) = p^{-r} N \phi^r_{D'}(v) = p^{s-r} Nv.$$

But $(s-r)/r = \alpha - 1$, so $Nv \in D'(\alpha - 1)$.

Hence, by descent from $\widehat{K_0^{ur}}$, we get $N(D(\alpha)) \subset D(\alpha - 1)$. But there are only finitely many α such that $D(\alpha) \neq 0$, the nilpotence of N now follows. □

Also notice that the definition of weak-admissibility applies to (ϕ, N)-modules well.

Lemma 6.2. If $0 \to D' \to D \to D'' \to 0$ is a short exact sequence in $\mathrm{MF}_K^{\phi,N}$ and any two of the three terms are weakly admissible then so is the third.

Proof. Omitted. □

Theorem 6.3. The category $\mathrm{MF}_K^{\phi,N}$ is abelian.

Proof. Omitted. □

Now, we want to construct the period ring B_{st}. Still, the semistable representations are meant to capture those with “bad reductions”, where it turns out that semistable reduction is the worst case. Here semistable reduction means it has worst normal crossing singularities, i.e., look like étale-locally like transverse intersections of hyperplanes in an affine space. For example, the elliptic curve has semistable reduction if and only if its reduction has a node as what we showed in 4.5.

The Tate curve $E_q = \mathbb{G}_m^{an}/q^{\mathbb{Z}}$ over K will tell us what we need: the p-adic Tate module representation has a $\mathbb{Z}_p$-basis given choices of $\epsilon = (\zeta_{p^n}) \in R$ and $\tilde{q} \in R$ satisfying $\tilde{q}^{(0)} = q \in K^*$, and the G_K-action relative to this basis is

$$\begin{pmatrix} \chi & \eta_{\tilde{q}} \\ 0 & 1 \end{pmatrix},$$

where $g(\tilde{q})/\tilde{q} = \epsilon^{\eta_{\tilde{q}}(g)}$ for a continuous 1-cocycle $\eta_{\tilde{q}} : G_K \to \mathbb{Z}_p$ relative to the χ-action.

References

[1] URL: https://ayoucis.wordpress.com/2014/11/29/classifying-one-dimensional-algebraic-groups/.

[2] URL: https://stacks.math.columbia.edu/tag/09ZS.

[3] Brain Conrad. *Semistable Reduction of Abelian Varieties*. URL: https://math.stanford.edu/~conrad/DarmonCM/2011Notes/SemistableReduction.pdf.

[4] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. 2nd. Vol. 106. Graduate Texts in Mathematics. Springer, 2009. DOI: 10.1007/978-0-387-09494-6.