SEMISTABLE HIGGS BUNDLES

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We would like to give an introduction on the theory of semistable Higgs bundles. Our final goal is to show the category of semistable Higgs bundles of same the slope is an abelian category, for which the author was unabble to find a reference. This is a generalization of the well-known fact that the category of semistable Higgs bundles of same slope is an abelian category.

Definition 0.1 (Higgs bundle)**.** Let X be a scheme over a field k. A Higgs bundle is defined to be a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a locally free sheaf over X and $\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ is an $\mathcal{O}_X$-linear map satisfying the integrality condition $\theta \wedge \theta = 0$. We also call such θ a Higgs field.

From now, fix a smooth proper curve C over a field k. Then the condition $\theta \wedge \theta$ will be automatic for any linear map $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_C$, as $\Omega^2_C = 0$.

Let $(\mathcal{E}_1, \theta_1), (\mathcal{E}_2, \theta_2)$ be two Higgs bundles on C, a morphism $f : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ is defined to be a $\mathcal{O}_C$-linear morphism (of sheaves) $f : \mathcal{E}_1 \to \mathcal{E}_2$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\quad f \quad} & \mathcal{E}_2 \\ \big\downarrow{\scriptstyle \theta_1} & & \big\downarrow{\scriptstyle \theta_2} \\ \mathcal{E}_1 \otimes \Omega^1_C & \xrightarrow[f \otimes 1]{} & \mathcal{E}_2 \otimes \Omega^1_C \end{array}$$

We now write Hig_C for the category of Higgs bundles over C/k.

Lemma 0.2. A coherent $\mathcal{O}_C$-module is locally free if and only if it is torsion-free.

Proof. Since $\dim C = 1$ and C is integral smooth, this lemma follows from [Sta25, Tag 0CC4]. □

Let $f : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ be a morphism of Higgs bundles. Let $\mathrm{Ker}(f)$ be the kernel of the morphism of sheaves $f : \mathcal{E}_1 \to \mathcal{E}_2$. $\mathrm{Ker}(f)$ is torsion-free as being a subsheaf of $\mathcal{E}_1$, and hence is locally free by Lemma 0.2. Moreover, since Ω^1_C is a locally free sheaf, we see $\mathrm{Ker}(f \otimes 1) \cong \mathrm{Ker}(f) \otimes 1$. In particular, $\theta_1 : \mathrm{Ker}(f) \to \mathrm{Ker}(f) \otimes \Omega_C$ is a well-defined morphism and $(\mathrm{Ker}(f), \theta_1)$ gives a Higgs bundle.

Similarly, define $\mathrm{Im}(f) := \mathrm{Im}(f : \mathcal{E}_1 \to \mathcal{E}_2)$. It is torsion-free sheaf as being a subsheaf of $\mathcal{E}_2$. $\mathrm{Im}(f)$ is then a locally free sheaf by 0.2 again. It is preserved by θ_2 as $\mathrm{Im}(f \otimes 1) = \mathrm{Im}(f) \otimes 1$. In particular, $(\mathrm{Im}(f), \theta_2)$ is a Higgs bundle.

Let $(\mathcal{E}, \theta)$ be a Higgs bundle. Define a Higgs subsheaf $(\mathcal{F}, \theta) \subset (\mathcal{E}, \theta)$ to be a pair consisting of a locally free subsheaf $\mathcal{F} \subseteq \mathcal{E}$ preserved by the Higgs field θ.

It is known that the Picard group $\mathrm{Pic}(C)$ admits a surjective homomorphism $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$. Let $\mathcal{E}$ be a vector bundle (a locally free sheaf) over C. Define its degree to be $\deg(\mathcal{E}) := \deg(\det(\mathcal{E}))$ and the slope to be

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rank}(E)}.$$

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Definition 0.3. Let $(\mathcal{E},\theta)$ be a Higgs bundle over C.

1. We say $(\mathcal{E},\theta)$ is semistable if for every sub-Higgs bundle $(\mathcal{F},\theta)\subseteq(\mathcal{E},\theta)$, i.e., $\mathcal{F}\subseteq\mathcal{E}$ is a locally free subsheaf preserving the Higgs field θ. Then one has

$$\mu(\mathcal{F})\leq\mu(\mathcal{E}).$$

2. We say $(\mathcal{E},\theta)$ is stable if for every sub-Higgs bundle $(\mathcal{F},\theta)\subseteq(\mathcal{E},\theta)$, i.e., $\mathcal{F}\subseteq\mathcal{E}$ is a locally free subsheaf preserving the Higgs field θ. Then one has

$$\mu(\mathcal{F})<\mu(\mathcal{E}).$$

3. We say $(\mathcal{E},\theta)$ is polystable if $(\mathcal{E},\theta)$ is a direct sum of stable Higgs bundles of the same slope.

In particular, if $\mathcal{E}$ is (semi)-stable in the classical sense, $(\mathcal{E},0)$ is a (semi)-stable Higgs bundle.

We note that the testing objects for (semi)-stability are subsheaves of $\mathcal{E}$. Some authors prefer to restrict on subbundles $\mathcal{F}\subseteq\mathcal{E}$, i.e., it also has torsion-free cokernel. We are claiming these two notions coincide for the curve case. One direction is obvious. Conversely, we need to show if $(\mathcal{E},\theta)$ is a Higgs bundle such that for any subbundle $\mathcal{F}\subset\mathcal{E}$ preserved by θ, then $\mu(\mathcal{F})<\mu(\mathcal{E})$ (resp. $\mu(\mathcal{F})\leq\mu(\mathcal{E})$). Then, for any locally free subsheaf $\mathcal{F}\subseteq\mathcal{E}$ preserved by θ, one still has $\mu(\mathcal{F})<\mu(\mathcal{E})$ (resp. $\mu(\mathcal{F})\leq\mu(\mathcal{E})$). In order to do this, we want to recall the saturation of a (Higgs) bundle.

Lemma 0.4. Let $\mathcal{E}$ be a vector bundle over C, and let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}$. Then

1. $\mathcal{F}$ is a vector bundle, and $\mathcal{F}$ is contained in a subbundle $\widetilde{\mathcal{F}}$ of $\mathcal{E}$ with $\operatorname{rank}(\mathcal{F})=\operatorname{rank}(\widetilde{\mathcal{F}})$ and $\deg(\mathcal{F})\leq\deg(\widetilde{\mathcal{F}})$.
2. If $(\mathcal{E},\theta)$ is a Higgs field with $\mathcal{F}$ preserved by θ, then $\widetilde{\mathcal{F}}$ is also preserved by θ.

Proof. 1. Since $\mathcal{F}$ is torsion-free as being a subsheaf of $\mathcal{E}$, it is locally free by Lemma 0.2. Let $\mathcal{T}$ be the torsion part of $\mathcal{E}/\mathcal{F}$ and take $\widetilde{\mathcal{F}}$ to be the preimage of $\mathcal{T}$ via the quotient map $\mathcal{E}\twoheadrightarrow\mathcal{E}/\mathcal{F}$. $\widetilde{\mathcal{F}}$ is again a vector bundle with $\mathcal{E}/\widetilde{\mathcal{F}}$ being torsion-free by construction. By Lemma 0.2 again, $\widetilde{\mathcal{F}}$ is then a subbundle of $\mathcal{E}$ containing $\mathcal{F}$, with $\widetilde{\mathcal{F}}/\mathcal{F}\cong\mathcal{T}$ is torsion. As $\mathcal{T}$ is torsion, it has rank 0. So, $\operatorname{rank}(\widetilde{\mathcal{F}})=\operatorname{rank}(\mathcal{F})$. The embedding $\mathcal{F}\hookrightarrow\widetilde{\mathcal{F}}$ also gives an embedding $\det(\mathcal{F})\hookrightarrow\det(\widetilde{\mathcal{F}})$, from where we have $\deg(\mathcal{F})\leq\deg(\widetilde{\mathcal{F}})$.

2. Consider the composition of maps

$$\widetilde{\mathcal{F}}\to\mathcal{E}\otimes\Omega_C^1\twoheadrightarrow(\mathcal{E}/\widetilde{\mathcal{F}})\otimes\Omega_C^1.$$

It obviously vanishes $\mathcal{F}$, and we hence have a $\mathcal{O}_C$-linear map

$$\mathcal{T}\to(\mathcal{E}/\widetilde{\mathcal{F}})\otimes\Omega_C^1.$$

By construction, $\mathcal{T}$ is torsion and $(\mathcal{E}/\widetilde{\mathcal{F}})\otimes\Omega_C^1$ is torsion-free, so such a map must be trivial. As a result, the $\mathcal{O}_C$-linear map $\widetilde{\mathcal{F}}\to(\mathcal{E}/\widetilde{\mathcal{F}})\otimes\Omega_C^1$ is trivial, and $\theta:\widetilde{\mathcal{F}}\to\widetilde{\mathcal{F}}\otimes\Omega_C^1$ is well-defined. □

Remark 0.5. In particular, let $(\mathcal{E},\theta)$ be a Higgs bundle such that for any subbundle $\mathcal{F}\subset\mathcal{E}$ preserved by θ, then $\mu(\mathcal{F})<\mu(\mathcal{E})$ (resp. $\mu(\mathcal{F})\leq\mu(\mathcal{E})$). Let $\mathcal{F}\subseteq\mathcal{E}$ be a subsheaf preserved by θ. There then exists a subbundle $\widetilde{\mathcal{F}}$ of $\mathcal{E}$ with the Higgs field θ by Lemma 0.4. Moreover, $\mathcal{F}\subseteq\widetilde{\mathcal{F}}$ by construction and $\mu(\widetilde{\mathcal{F}})<\mu(\mathcal{E})$ (resp. $\mu(\widetilde{\mathcal{F}})\leq\mu(\mathcal{E})$) by the stability (resp. semi-stability) assumption on $(\mathcal{E},\theta)$. So, we could draw the conclusion that the testing objects could just be taken to be subbundles.

Lemma 0.6. Let $\mathcal{E},\mathcal{F},\mathcal{G}$ be vector bundles over C. Assume there exists a short exact sequence

$$0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0.$$

1. We have

$$\operatorname{rank}(\mathcal{F})=\operatorname{rank}(\mathcal{E})+\operatorname{rank}(\mathcal{G}),\deg(\mathcal{F})=\deg(\mathcal{E})+\deg(\mathcal{G}).$$

2. If $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are all nonzero, then we have

$$\min\{\mu(\mathcal{E}), \mu(\mathcal{G})\} \leq \mu(\mathcal{F}) \leq \max\{\mu(\mathcal{E}), \mu(\mathcal{G})\},$$

with either equality if and only if $\mu(\mathcal{E}) = \mu(\mathcal{G})$.

Proof. The first claim is easy. For the second one, we note that

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} = \frac{\deg(\mathcal{E}) + \deg(\mathcal{G})}{\operatorname{rank}(\mathcal{E}) + \operatorname{rank}(\mathcal{G})}.$$

The second claim should follow from an easy calculation. □

Theorem 0.7. Let $(\mathcal{E}_1, \theta_1), (\mathcal{E}_2, \theta_2)$ be two semistable Higgs bundles. Then

$$\operatorname{Hom}_{\operatorname{Hig}_C}((\mathcal{E}_1, \theta_1), (\mathcal{E}_2, \theta_2)) = 0$$

if $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_1)$.

Proof. Assume $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_1)$ and $f : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ is a nonzero morphism. So, $(\operatorname{Im}(f), \theta_2) \subseteq (\mathcal{E}_2, \theta_2)$ is a nonzero Higgs subsheaf and

$$\mu(\operatorname{Im}(f)) \leq \mu(\mathcal{E}_2) < \mu(\mathcal{E}_1) \tag{$*$}$$

by assumption.

On the other hand, we have a short exact sequence of Higgs bundles

$$0 \to (\operatorname{Ker}(f), \theta_1) \to (\mathcal{E}_1, \theta_1) \to (\operatorname{Im}(f), \theta_2) \to 0.$$

$\operatorname{Ker}(f)$ cannot be zero, otherwise $(\mathcal{E}_1, \theta)$ will be a Higgs subsheaf of $(\mathcal{E}_2, \theta_2)$ and contradicts the semistability of $(\mathcal{E}_2, \theta_2)$. By assumption, $\mu(\operatorname{Ker}(f)) \leq \mu(\mathcal{E}_1)$ and hence

$$\mu(\mathcal{E}_1) \leq \max\{\mu(\operatorname{Ker}(f)), \mu(\operatorname{Im}(f))\} = \mu(\operatorname{Ker}(f))$$

by $(*)$ and Lemma 0.6. It follows that $\mu(\mathcal{E}_1) = \mu(\operatorname{Ker}(f)) = \mu(\operatorname{Im}(f))$ by Lemma 0.6, which is in contradiction with $(*)$.

□

Corollary 0.8. Let $\mathcal{E}_1, \mathcal{E}_2$ be two semistable vector bundles over C. Then $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{E}_1, \mathcal{E}_2) = 0$ if $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_1)$.

Theorem 0.9 (Harder-Narasimhan filtration)**.** Let $\mathcal{E}$ be a vector bundle on C, then it admits a unique filtration by subbundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that the successive quotients $\mathcal{V}_i/\mathcal{V}_{i-1}, 1 \leq i \leq n$ are semistable vector bundles over C with

$$\mu(\mathcal{E}_1/\mathcal{E}_0) > \cdots > \mu(\mathcal{E}_n/\mathcal{E}_{n-1}).$$

This filtration is called the Harder-Narasimhan filtration, which we sometimes also write as HN-filtration.

Define $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E}_1)$ and $\mu_{\min}(\mathcal{E}) = \mu(\mathcal{E}/\mathcal{E}_{n-1})$. Then one can has a generalization of Theorem 0.7 for vector bundles.

Theorem 0.10. Let $\mathcal{E}, \mathcal{F}$ be two vector bundles over C with $\mu_{\min}(\mathcal{E}) > \mu_{\max}(\mathcal{F})$. Then

$$\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{F}) = 0.$$

Proof. Suppose $\mu_{\min}(\mathcal{E}) > \mu_{\max}(\mathcal{F})$ and $\phi : \mathcal{E} \to \mathcal{F}$ is a nontrivial morphism. Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

be the HN-filtration of $\mathcal{E}$ and

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$$

be the HN-filtration of $\mathcal{F}$. Let $i \geq 1$ be the smallest number such that $\phi(\mathcal{E}) \subset \mathcal{F}_i$. Then ϕ gives a nontrivial morphism $\phi : \mathcal{E} \to \mathcal{F}_i \twoheadrightarrow \mathcal{F}_i/\mathcal{F}_{i-1}$, with the target is semistable of slope $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) < \mu_{\max}(\mathcal{F})$. On the other hand, let $j \geq 1$ be the smallest number such that $\phi : \mathcal{E}_j \to \mathcal{F}_i/\mathcal{F}_{i-1}$ is nontrivial. It then gives a nontrivial morphism $\phi : \mathcal{E}_j/\mathcal{E}_{j-1} \to \mathcal{F}_i/\mathcal{F}_{i-1}$, with $\mathcal{E}_j/\mathcal{E}_{j-1}$ semistable of slope $\mu(\mathcal{E}_j/\mathcal{E}_{j-1}) > \mu_{\min}(\mathcal{E}) > \mu_{\max}(\mathcal{F}) > \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$. Then we have a contradiction by Theorem 0.7. □

Remark 0.11. One should not expect Theorem 0.10 to work for Higgs bundles, as the HN filtration is not stable under the Higgs field in general.

Lemma 0.12. Let $(\mathcal{E}, \theta)$ be a Higgs stable (resp. semistable) vector bundle. Then the Higgs bundle $(\mathcal{E}^\vee, \theta')$ is still Higgs stable (resp. semistable), where θ' is given by taking the dual of θ and then tensoring with $\Omega_{C/k}$ on both sides.

Proof. Let $(\mathcal{E}, \theta)$ be a Higgs stable (resp. semistable) vector bundle over C. It suffices to show if $(\mathcal{F}, \theta') \subseteq (\mathcal{E}^\vee, \theta')$ is a subbundle, then $\mu(\mathcal{F}) < \mu(\mathcal{E}^\vee)$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E}^\vee)$) by Remark 0.5.

As $\mathcal{F} \subseteq \mathcal{E}^\vee$ and is a subbundle preserved by θ, $(\mathcal{E}^\vee/\mathcal{F}, \theta')$ is also a Higgs bundle and one has a short exact sequence

$$0 \to (\mathcal{F}, \theta') \to (\mathcal{E}^\vee, \theta') \to (\mathcal{E}^\vee/\mathcal{F}, \theta') \to 0.$$

Taking dual and tensoring with Ω_C again, one has a short exact sequence

$$0 \to ((\mathcal{E}^\vee/\mathcal{F})^\vee, \theta) \to (\mathcal{E}, \theta) \to (\mathcal{F}^\vee, \theta) \to 0.$$

As $(\mathcal{E}, \theta)$ is Higgs stable (resp. Higgs semistable), one has $\mu((\mathcal{E}^\vee/\mathcal{F})^\vee) < \mu(\mathcal{E})$ (resp. $\mu((\mathcal{E}^\vee/\mathcal{F})^\vee) \leq \mu(\mathcal{E})$). So, $\mu(\mathcal{E}^\vee/\mathcal{F}) > \mu(\mathcal{E})$ (resp. $\mu(\mathcal{E}^\vee/\mathcal{F}) \geq \mu(\mathcal{E})$). Lemma 0.6 then implies $\mu(\mathcal{F}) < \mu(\mathcal{E}^\vee)$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E}^\vee)$), which finishes the proof.

□

Lemma 0.13 (Destabilizing submodule)**.** Let $(\mathcal{E}, \theta)$ be a Higgs bundle and suppose it is not Higgs semistable. Then there exists a Higgs subsheaf $(\mathcal{F}_{\max}, \theta) \subset (\mathcal{E}, \theta)$ such that for any Higgs subsheaf $(\mathcal{F}, \theta) \subset (\mathcal{E}, \theta)$, one has $\mu(\mathcal{F}) \leq \mu(\mathcal{F}_{\max})$. With equality only if $\mathcal{F}_{\max} = \mathcal{F}$. This is called the maximal destabilizing subsheaf, and is unique up to isomorphism.

Proof. Consider the set

$$S = \{\mu(\mathcal{F}); (\mathcal{F}, \theta) \subset (\mathcal{E}, \theta) \text{ is a Higgs subsheaf and } \mu(\mathcal{F}) > \mu(\mathcal{E})\}.$$

S is not empty as $(\mathcal{E}, \theta)$ is not Higgs semistable. S is a discrete subset of $\mathbb{Q}$, as $\mathcal{F} \subseteq \mathcal{E}$ implies $\deg(\mathcal{F}) \leq \deg(\mathcal{E})$. Let $\max(S)$ be the supremum of S and define

$$S' := \{(\mathcal{F}, \theta); \mu(\mathcal{F}) = \max(S)\}$$

to be set of Higgs subsheaves reaching the maximum slope. Note that as the saturation $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ in $\mathcal{E}$ satisfies $\mu(\widetilde{\mathcal{F}}) \geq \mu(\mathcal{F})$, S' acturally consists only of Higgs subbundles. S' also has a partial order given by $(\mathcal{F}, \theta) \leq (\mathcal{F}', \theta)$ if $\mathcal{F} \subseteq \mathcal{F}'$. Let $(\mathcal{F}_{\max}, \theta)$ be a maximal element of S'. We now wish to prove this is unique. If not, let $(\mathcal{F}', \theta)$ be another maximal element in S'. Consider the natural map of Higgs bundles $\phi : (\mathcal{F}_{\max} \oplus \mathcal{F}', \theta \oplus \theta) \to (\mathcal{E}, \theta)$. It gives a short exact sequence

$$0 \to (\mathcal{F}_{\max} \cap \mathcal{F}', \theta) \to (\mathcal{F}_{\max} \oplus \mathcal{F}', \theta \oplus \theta) \to (\mathrm{Im}(\phi), \theta) \to 0.$$

By construction, one has $\mu(\mathcal{F}_{\max} \oplus \mathcal{F}') = \mu_{\max}$ and $\mu(\mathrm{Im}(\phi)), \mu(\mathcal{F}_{\max} \cap \mathcal{F}') \leq \mu_{\max}$. Lemma 0.6 then implies that $\mu(\mathcal{F}_{\max} \cap \mathcal{F}') = \mu(\mathrm{Im}(\phi)) = \mu_{\max}$. On the other hand, $\mathrm{Im}(\phi)$ can be defined as the sheafification of the presheaf $U \mapsto \mathcal{F}_{\max}(U) + \mathcal{F}'(U)$. As the sheafification functor is exact, one has $\mathcal{F}', \mathcal{F}_{\max} \subseteq \mathrm{Im}(\phi)$. By the maximality of $\mathcal{F}'$ and $\mathcal{F}_{\max}$, one must have $\mathcal{F}' = \mathrm{Im}(\phi) = \mathcal{F}_{\max}$ and the lemma follows.

□

Lemma 0.14. Let $(\mathcal{E}, \theta)$ be a Higgs bundle over C. Let $f : D \to C$ be an étale cover of degree d between irreducible smooth proper curves over k. Then $(\mathcal{E}, \theta)$ is Higgs stable (resp. Higgs semistable) if and only if $(f^*\mathcal{E}, f^*\theta)$ is Higgs stable (resp. Higgs semistable).

Proof. Since f is étale, one has $f^*\Omega_C \cong \Omega_D$ and hence $(f^*\mathcal{E}, f^*\theta)$ is well-defined. Suppose $(f^*\mathcal{E}, f^*\theta)$ is Higgs stable (resp. Higgs semistable). Let $(\mathcal{F}, \theta) \subset (\mathcal{E}, \theta)$ be a Higgs subbundle. Then it gives $(f^*\mathcal{F}, f^*\theta) \subset (f^*\mathcal{E}, f^*\theta)$ as a Higgs subbundle. So, one has $\mu(f^*\mathcal{F}) < \mu(f^*\mathcal{E})$ (resp. $\mu(f^*\mathcal{F}) \leq \mu(f^*\mathcal{E})$). Since for any vector bundle $\mathcal{V}$ over C, $\mathrm{rank}(f^*\mathcal{V}) = \mathrm{rank}(\mathcal{V})$ and $\deg(f^*\mathcal{V}) = d \cdot \deg(\mathcal{V})$, the "if" direction follows.

Conversely, let $C' \to C$ be a Galois cover such that $C' \to D$ is also an étale cover. Then it suffices to show the "only if" direction when $D \to C$ is a Galois cover, as we already showed the "if" direction. Let $G = \mathrm{Aut}(D/C)$. And let $(\mathcal{F}', \theta') \subset (f^*\mathcal{E}, f^*\theta)$ be the maximal destabilizing subsheaf. Since the maximal destabilizing subsheaf is unique by Lemma 0.13, $(\mathcal{F}', \theta')$ is stable under the action of G given by the descent datum of $(f^*\mathcal{E}, f^*\theta)$. By étale descent, there exists a subsheaf $\mathcal{F} \subset \mathcal{E}$ over C such that $f^*\mathcal{F} \cong \mathcal{F}'$. Moreover, $\theta' : \mathcal{F}' \to \mathcal{F}' \otimes f^*\Omega_C$ can also be descended to $\theta : \mathcal{F} \to \mathcal{F} \otimes \Omega_C$ as it is preserved by the descent diagram given by $f^*\theta$. Finally, as $(\mathcal{E}, \theta)$ is Higgs stable (Higgs semistable), it follows that $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$). The lemma then follows from the calculation that $\mu(f^*\mathcal{E}) = d \cdot \mu(\mathcal{E}), \mu(f^*\mathcal{F}) = d \cdot \mu(\mathcal{F})$. □

Lemma 0.15. Let $\mathcal{E}$ be a vector bundle over C and let $f : D \to C$ be an étale cover between smooth proper curves. Then,

1. The HN-filtration of $\mathcal{E}^\vee$ satisfies $\mu_{\min}(\mathcal{E}^\vee) = -\mu_{\max}(\mathcal{E})$ and $\mu_{\max}(\mathcal{E}^\vee) = -\mu_{\min}(\mathcal{E})$.
2. If the HN-filtration of $\mathcal{E}$ is

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E},$$

Then the HN-filtration of $f^*\mathcal{E}$ is given by

$$0 = f^*\mathcal{E}_0 \subset f^*\mathcal{E}_1 \subset \cdots \subset f^*\mathcal{E}_n = f^*\mathcal{E}.$$

Proof. 1. Let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$ be the HN-filtration of $\mathcal{E}$. For each $0 \leq i \leq n$, consider the short exact sequence of vector bundles

$$0 \to \mathcal{E}_i \to \mathcal{E} \to Q_i \to 0,$$

with $Q_i \cong \mathcal{E}/\mathcal{E}_i$. Taking the dual and we have

$$0 \to Q_i^\vee \to \mathcal{E}^\vee \to \mathcal{E}_i^\vee \to 0.$$

Consider the filtration of $\mathcal{E}^\vee$

$$0 = Q_n^\vee \subset Q_{n-1}^\vee \subset \cdots \subset Q_0^\vee = \mathcal{E}^\vee.$$

Then $Q_i^\vee/Q_{i+1}^\vee \cong (\mathcal{E}_{i+1}/\mathcal{E}_i)^\vee$ is semistable of slope $-\mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ by Lemma 0.12. The statement then follows by the inequality

$$\mu(Q_i^\vee/Q_{i+1}^\vee) = -\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) < -\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(Q_{i-1}^\vee/Q_i^\vee).$$

2. This is obvious by Lemma 0.14, as f^* is exact and $\mu(f^*\mathcal{E}) = \deg(f) \cdot \mu(\mathcal{E})$ for any vector bundle $\mathcal{E}$ over C. □

Theorem 0.16. Let $f : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ be a morphism of Higgs semistable bundles of the same slope μ. Then one has

1. The kernel, image and cokernel of f will all be either trivial or Higgs semistable bundles of slope μ.

2. Let $(\mathcal{E}_1,\theta_1),(\mathcal{E}_3,\theta_3)$ be Higgs semistable bundles of the same slope μ and suppose we have a short exact sequence of Higgs bundles

$$0 \to (\mathcal{E}_1,\theta_1) \to (\mathcal{E}_2,\theta_2) \to (\mathcal{E}_3,\theta_3) \to 0,$$

then $(\mathcal{E}_2,\theta_2)$ is Higgs semistable of slope μ.

Proof. 1. Consider the short exact sequence of Higgs bundles

$$0 \to (\mathrm{Ker}(f),\theta_1) \to (\mathcal{E}_1,\theta_1) \to (\mathrm{Im}(f),\theta_1) \to 0.$$

Let $\widetilde{\mathrm{Im}(f)}$ be the saturation of $\mathrm{Im}(f)$ in $\mathcal{E}_2$ and we have a Higgs bundle $(\widetilde{\mathrm{Im}(f)},\theta_2)$ by Lemma 0.4. Consider the short exact sequence

$$0 \to (\widetilde{\mathrm{Im}(f)},\theta_2) \to (\mathcal{E}_2,\theta_2) \to (Q,\theta_2) \to 0.$$

As $(\mathcal{E}_1,\theta_1),(\mathcal{E}_2,\theta_2)$ are both Higgs semistable of slope μ, one has

$$\mu(\mathrm{Ker}(f)) \le \mu(\mathcal{E}_1) = \mu,$$

$$\mu(\mathrm{Im}(f)) \le \mu(\widetilde{\mathrm{Im}(f)}) \le \mu(\mathcal{E}_2) = \mu.$$

By Lemma 0.6, one must have $\mu(\mathrm{Ker}(f)) = \mu(\mathrm{Im}(f)) = \mu(\widetilde{\mathrm{Im}(f)}) = \mu$. In particular, one has $\mathrm{Im}(f) \cong \widetilde{\mathrm{Im}(f)}$ and hence $\mathrm{Im}(f)$ is a subbundle. It implies the cokernel $(\mathrm{Coker}(f),\theta_2) \cong (Q,\theta_2)$ is a Higgs bundle and Lemma 0.6 again implies that $\mu(Q) = \mu$. $(\mathrm{Ker}(f),\theta_1)$ and $(\mathrm{Im}(f),\theta)$ are Higgs semistable obviously. It remains to show $(\mathrm{Coker}(f),\theta_2)$ is Higgs semistable. Let $(\mathcal{F},\theta_2) \subset (\mathrm{Coker}(f),\theta_2)$ be a Higgs subbundle. Let $\widetilde{\mathcal{F}}$ be the preimage via the surjection $\mathcal{F}$ of $\mathcal{E}_2 \twoheadrightarrow \mathrm{Coker}(f)$. $\widetilde{\mathcal{F}}$ is then preserved by θ_2. In particular, we have a short exact sequence

$$0 \to (\mathrm{Im}(f),\theta_2) \to (\widetilde{\mathcal{F}},\theta_2) \to (\mathcal{F},\theta_2) \to 0.$$

As $(\mathcal{E}_2,\theta_2)$ is Higgs semistable of slope μ, $\mu(\widetilde{\mathcal{F}}) \le \mu$. And $\mu(\mathcal{F}) \le \mu$ follows from Lemma 0.6. So, $(\mathrm{Coker}(f),\theta_2)$ is Higgs semistable.

2. $\mu(\mathcal{E}_2) = \mu$ follows from Lemma 0.6. We wish to show $(\mathcal{E}_2,\theta_2)$ is Higgs semistable. Let $(\mathcal{F},\theta) \subset (\mathcal{E}_2,\theta_2)$ be a Higgs subbundle. Then $(\mathcal{F}\cap\mathcal{E}_1,\theta)$ gives a Higgs subsheaf of $(\mathcal{F},\theta)$. As there is an injection $\mathcal{F}/\mathcal{F}\cap\mathcal{E}_1 \hookrightarrow \mathcal{E}_2/\mathcal{E}_1 \cong \mathcal{E}_3$ with target torsion-free, one has $(\mathcal{F}\cap\mathcal{E}_1,\theta)$ is a Higgs subbundle of $(\mathcal{F},\theta)$. Consider the short exact sequence

$$0 \to (\mathcal{F}\cap\mathcal{E}_1,\theta) \to (\mathcal{F},\theta) \to (\mathcal{F}/\mathcal{F}\cap\mathcal{E}_1,\theta_1) \to 0.$$

As $(\mathcal{E}_1,\theta_1),(\mathcal{E}_3,\theta_3)$ are Higgs semistable of slope μ. One has $\mu(\mathcal{F}) \le \mu$ and $\mu(\mathcal{F}/\mathcal{F}\cap\mathcal{E}_1) \le \mu$. By Lemma 0.6, $\mu(\mathcal{F}) \le \mu$ also and it follows that $(\mathcal{E}_2,\theta_2)$ is Higgs semistable. □

Corollary 0.17. The category of Higgs semistable bundles of fixed slope μ is an abelian category.

Corollary 0.18. A Higgs polystable bundle is Higgs semistable.

References

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